

Tensorial Quantum Gravity and the Cosmological Constant Problem

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I dedicate this work to Odilia.

Abstract

In the present article, which is the first part of a work in three parts, we build an equation of quantum gravity. This equation is tensorial, is equivalent to general relativity in vacuum, but differs completely from general relativity inside matter. We can spot directly in the equation the terms representing the perturbative quantum corrections to classical gravity and the nonperturbative quantum corrections. This new equation possesses a dimensionless gravitational coupling constant, and passes all the experimental tests that also passes general relativity, because concerning these tests, the predictions of both theories are identical. This quantum gravity and general relativity diverge essentially in the domain of cosmology : we prove that quantum gravity gives the solution to the whole set of problems left over by the standard cosmological model based on general relativity. Essentially we prove that the initial singularity, the big bang, is smoothed out by quantum gravity, that the flatness problem finds a precise solution : quantum gravity predicts that Ω should be just a little more than 1, which fits perfectly with the observed value, which is around 1.02. The cosmological constant problem also finds its solution since we prove that the Λ term does not come from any dark energy, but comes from nonperturbative quantum corrections to classical relativity, and has the exact tiny but strictly positive value needed. Furthermore, the quantum equation of gravity possesses with no further efforts features of unification. Indeed, our equation governs at the same time the large scale of the universe, as did general relativity, but also the structure of particles.

Part I

The quantum equation of gravity

1 Introduction

1.1 Successes and drawbacks of general relativity

Would we like to make some comments very much at random on the successes and drawbacks of general relativity, we first would emphasize that on one hand this theory passed all experimental tests, and on the other hand, from a theoretical point of view, we notice that this theory resists much to being quantized by the methods in vigor to quantize gauge theories, which describe the three other interactions. It is especially well known that a theory, to be renormalizable, must have a dimensionless coupling constant and that this is not the case of general relativity.

1.2 General relativity in vacuum

By having a closer look at general relativity, we notice that the experimental tests which are known to confirm this theory in fact only test the equation $R_{ik} = 0$, or equivalently only test the theory in vacuum. The reason for this is that the calculations of trajectories of massive objects placed in a gravitational field, in the context of general relativity, use only the equation of vacuum $R_{ik} = 0$, eventually up to the computation of the masses that generate this gravitational field. When we use only the equation $R_{ik} = 0$, these masses appear as constants of integration. In other words, the calculation of the deviation of a beam of light which passes near the sun, but still propagates in vacuum, the calculation of the advance in the perihelion of the planet Mercury, which propagates in vacuum too, the calculations concerning gravitational waves propagating in vacuum and the approximation to very small fields are all based on the sole $R_{ik} = 0$.

1.3 General relativity inside matter

So if the experimental tests designed to test general relativity only test the vacuum equation $R_{ik} = 0$, in which domain of physics do we effectively use and verify experimentally the equations of general relativity inside matter? Because general relativity is a theory based on inertia, and because inertia can only be defined in vacuum, only general relativity in vacuum is needed to study physics in a gravitational field, once given this field. But to compute the masses that generate this gravitational field, we suppose that the celestial bodies are filled in by a uniform energy density ϵ , and we use the Schwarzschild solution to compute these masses : this sole calculation, at the same time, proves that Newton's theory is retrieved as a limit of general relativity, and determines the coupling constant of general relativity relatively to Newton's constant. So, concerning this problem, if we use only general relativity in vacuum we are able to do all what we can do with general relativity except that we cannot compute the masses of the celestial bodies, and that we cannot retrieve Newton's theory. Or more precisely, we can retrieve Newton's theory, but we have to put by hand the masses in our calculations. This problem of computing the masses will find in our construction a solution directly from the calculation of the masses of the fundamental particles, as in our second article [56]. For the time being, we can leave this problem and look towards the other domains of physics where the equations of general relativity inside matter come into play. As far as black holes are concerned, what happens inside a black hole is not easily confronted to experiment. So in fact the only domain where these equations inside matter are really used in physics and where a possible confrontation to experiment can be made is cosmology. So, when we arrive at the conclusion that we should use the standard cosmological model to probe the equations of general relativity inside matter, we should not forget either that this must be done keeping in mind that these equations have never passed so far any experimental test : we can be assured of general relativity, but only in vacuum. We believe this is the key explanation of why gravity has been so difficult to quantize : we trusted general relativity inside matter because of its successes in vacuum, but nothing prevents the existence of another theory, identical to general relativity in vacuum, but differing from it inside matter, and for example with a dimensionless coupling constant. We now return to our main

idea, that general relativity inside matter should be experimentally checked by cosmology.

1.4 The standard cosmological model

The standard cosmological model possesses great successes and generate great difficulties too. The computation of the abundances of the elements, and of the cosmic background radiation, are successes which seem to indicate that our universe comes from a hot phase. We find the right abundances provided we suppose a great deal of non baryonic dark matter in the universe, and concerning the cosmic background, the behavior of the temperature $T \sim 1/a$ is a key argument to prove the observed perfect black body behavior of this radiation. But the flatness problem, the expansion problem, the problem of the initial singularity, the cosmological constant problem, and the need for so much dark matter to fit experiment are serious drawbacks of the model. We emphasize here that the cosmological inflationary models (Guth 1980 [21], Guth 1981 [22], Kazanas 1980 [30], Peebles 1993 [49], Sato 1981 [59], [60]), if they solve the flatness problem, do not solve to our opinion the initial singularity problem. This because the initial singularity problem is linked to the beginning of the universe. The fact is that the origin of the universe, the so called big bang, if it exists, needs a former cause. And there can be no former cause of the beginning of the universe. Now, this means exactly that the universe never began, which is one of the conclusions of the inflationary models. But in these models, the absence of a beginning is only due to one scalar particle, and we believe that such a contingent reality cannot be used to decide of the absence or existence of the beginning of the universe. Furthermore, this scalar particle is necessarily inside the universe and its existence is already one of the consequences of the existence of the universe, so this particle cannot be held to be the cause of the absence of the beginning of the universe if it is at the same time a consequence of its existence. Said differently, from a logical point of view, the absence or presence of a beginning of the universe is former to its existence, which is former to the existence of the particle, which for these reasons cannot be former to the absence of the beginning. In conclusion, we need to prove the absence of beginning by a fundamental law of nature, we mean by quantum gravity.

1.5 Supersymmetry, string theory and dualities

On the theoretical side, general relativity has become disastrous when physicists have begun to quantize it. The gravitational coupling constant is not even dimensionless. Supersymmetry, string theory and all its avatars born from the dualities, have as a common origin, the necessity to eliminate all the infinities left over by the quantization of general relativity. These theories also have two other common features. First they are very speculative and it is difficult to check their predictions by experimental tests, and second they are mathematically involved, especially because they consist in a try of a mathematical unification of the interactions. But in this context, unification has to be made like at random, exactly because experiment cannot guide us anymore, nor correct our hypotheses. The same happens with dark matter, some particles are predicted for example by supersymmetry which are not directly observed, and we have to hope that these particles will fill the gap between the known baryonic matter and the calculated total energy density in the universe. But still, from the hypothesis of the existence of these particles coming from supersymmetry to the calculation of the density of non baryonic dark matter, there are large regions of theoretical reasonings in which experiment cannot enter, as it should, to make physics go upon a much more secure path.

1.6 A tensorial equation of quantum gravity

As we said general relativity has misled us inside matter because of its successes in vacuum, in such a way that we did not see its failures. In fact, thinking of the equations of general relativity inside matter, we immediately notice a specificity of this theory : as it links the curvature tensor coming from gravity to the momentum-energy tensor of matter, included the momentum-energy tensor of electromagnetism, the theory contains necessarily implicit hypotheses on unification itself. We mean information about the way the different interactions behave themselves in respect to each other, and how they operate on fermions. And if the tensorial equation of gravity goes so far, it probably contains implicit hypotheses on quantum gravity itself. And this because the implicit hypotheses made on unification we were just talking about will have no chance to be right

if even quantum gravity is not taken into account. Now this has the following consequence : There must exist a tensorial equation describing not only classical gravity, but also its quantum corrections, so there exists a tensorial equation of quantum gravity which has furthermore all probabilities to lead us directly to unification.

1.7 Quantum gravity built out of experiment

Contrarily to what happens in string theory, we can build the equation of quantum gravity and check at each step of the construction that it is compatible with experiment. First, if the experimental tests of general relativity only test the equation in vacuum, we can construct this quantum theory out of general relativity by changing only the theory inside matter, in such a way that both theories coincide in the vacuum limit $\epsilon \rightarrow 0$. This way quantum gravity will pass the same experimental tests that passed general relativity. Furthermore we know that this quantum gravity should also predict the existence of a hot universe, and in some way should keep unchanged the standard calculations for the abundances of the elements and the cosmic background radiation. This way, all the successes of the standard cosmological model will be preserved. We also have the constraints and indications coming from the quantum regime of the other interactions. So we should construct a quantum equation of gravity which possesses a dimensionless coupling constant, and we should be able to read on this tensorial equation the perturbative and non perturbative quantum corrections to classical gravity. We also use the fact that cosmology is the preferred domain of application of tensorial gravity inside matter, and we now use at our advantage the drawbacks of the standard cosmological model. For this we adopt a minimal principle : the quantum equation of gravity should solve at once all the cosmological problems left over by general relativity with no additional hypothesis such as dark energy or non baryonic dark matter. So our simple idea, which turns out to become a method of investigation, is to construct quantum gravity as a tensorial equation, by simply correcting general relativity inside matter, such that all the successes of general relativity are conserved, and all the problems left over by this theory are automatically solved, with no additional hypothesis. This way, all experimental observations made in

the domain of cosmology turn out to be experimental indications on how quantum gravity should be built, and we can go further upon this secure path which later will lead us to a unified theory.

1.8 The specificity of quantum gravity

Once we make the necessary quantum corrections to classical general relativity, we find that as we had predicted, this quantum gravity will automatically contain unification. Indeed we prove in the present article that the quantum equations automatically determine the value of the pressure p which itself controls the structure of matter, relativistic or non relativistic. For example the quantum equation predicts for the early universe the value $p = \epsilon/3$ which has to be put by hand in the standard cosmological model. So the quantum equation has a first essential feature of unification : it governs at the same time the large scale and small scale of the universe.

1.9 The predictions of quantum gravity

At the same time, we are going to see in the present article how the quantum equation automatically gives a solution to the following cosmological problems. First the flatness problem : observations tend to give us a value of Ω around 1, but more precisely $\Omega = 1.02 \pm 0.02$ (Bennett and al. 2003 [2]). Quantum gravity predicts a value $\Omega \approx 1$ but also $\Omega > 1$. So not only gives quantum gravity the right value of $\Omega \approx 1$, but also it gives account for the fact that Ω has been observed to be a little greater than 1. Quantum gravity gives a solution to the expansion problem, because it predicts that $\ddot{a} > 0$. Finally quantum gravity gives a solution to the cosmological constant problem. It gives the origin of the Λ term in the equation : it is the term corresponding to the non perturbative quantum corrections to classical gravity, and also it predicts that this term acts as if, put by hand in the equation as it is in general relativity, it makes us see the total energy density of matter in the universe, the apparent energy density ϵ_{app} , multiplied by the factor 4, compared to what this energy density ϵ really is. We thus have the relation

$\epsilon_{app} = 4\epsilon$. This factor 4 fits strikingly with the observations of Bennett and al. 2003, [2], which give $\Omega = 1.02 \pm 0.02$ and $\Omega_\Lambda = 0.73 \pm 0.04$. In fact, we have to notice that these observations are model dependent. Furthermore, in quantum gravity, the energy density ϵ and the term Ω are not proportional, and we also find :

$$\Omega_\Lambda = \frac{1}{2}\Omega \tag{1.1}$$

which still solves the cosmological constant problem anyway, since it gives for the value of Λ the strictly positive and incredibly tiny value we were looking for. As a final remark, we add that quantum gravity, in fact, leads to an entire new class of cosmological models, quite near each other, and which all display the same general quantum features. It is then easy to change the coefficient 1/2 for another factor, provided we go from a quantum model to another.

1.10 Computing the masses

Our quantum theory in vacuum is equivalent to general relativity in vacuum. We said in section 1.3 that this situation makes quantum gravity and general relativity pass the same experimental tests, except that to retrieve Newton's theory with quantum gravity, we have to put the masses by hand. In fact, this will turn at the advantage of quantum gravity, because general relativity is helpless anyway to compute the masses of the fundamental particles, whereas quantum gravity will lead to the web of formulas which were needed to compute the parameters of the standard model of particle physics. This will be the subject of our second article, and we just explain here rapidly how these things are working. To describe a celestial body, there are three models, the classical, the quantum and the intermediate. The classical model is adapted to tensorial equations and thus to general relativity : celestial bodies are made of continuous, uniform energy densities. The quantum model contradicts by itself tensorial equations and especially general relativity because celestial bodies are made of pointlike particles, and the fields appearing in Einstein's equations can only take two extreme values : zero quasi everywhere, and infinity at a finite number of points, where particles are. Clearly we need a third and intermediate

model : a particle is a sphere of very small radius r and huge energy density ϵ . The intermediate model is powerful enough to contain both other models. The classical model is only a approximation of the intermediate model which is far more precise. This intermediate model contains the quantum model as a limit, when $r \rightarrow 0$, $\epsilon \rightarrow +\infty$, the mass of the particle being kept fixed. Our conclusion is thus the following : we can suppose that as far as general relativity inside matter is concerned, gravity couples to matter only inside particles. So it appears that in the context of our quantum equation of gravity, the only masses that we really have to put by hand are the masses of the particles. We will see in [56] how they can be computed, but we give here an example of how, from the point of view of unification, general relativity can be improved by quantum gravity to make possible the calculation of the parameters of the standard model.

1.11 A dimensionless quantum gravitational coupling around unity

We know that as far as the three other interactions are concerned, they are governed by gauge theories, and the coupling constants are dimensionless and around unity, which makes arise the question to know why gravitation is so tiny. The equations of general relativity inside matter, we mean their right hand side, possess a relativistic gravitational coupling constant κ which is linked to Newton's constant G by the formula :

$$\kappa = \frac{8\pi G}{c^4} \tag{1.2}$$

Now we recall that after renormalization, the coupling constants of the other interactions are not only dimensionless and around unity, but also depend on energy. So if we suppose for an instant that the quantum gravitational coupling also depends on energy as it should, in such a tensorial equation it would depend on the energy density. Since we look for a dimensionless constant we take

$$\kappa = \frac{8\pi G}{c^4} = \frac{\kappa_0}{\sqrt{\epsilon}} \tag{1.3}$$

Then appear two facts. First κ_0 has become dimensionless, and second and above all the value of κ_0 should be around unity. Indeed, we proved in section 1.10 that we can place

ourselves in the case of the intermediate model since this model contains the others, and furthermore we proved that in this model gravity couples to matter only inside particles, so the value of ϵ in (1.3) should be taken to be the energy density ϵ_P inside particles. Now what could be the energy density inside a particle? It should be the greatest energy density we can think of, which is the energy density corresponding to the Planck mass in every sphere of radius the Planck length. We thus find

$$\epsilon_P = \frac{3M_P}{4\pi L_P^3} \approx \frac{1}{L_P^4} = G^{-2} \quad (1.4)$$

by the definition of ϵ_P , where we left the coefficient $4\pi/3$ of the volume of a sphere because it is around unity, where $M_P = 1/L_P$, with M_P and L_P respectively the Planck mass and the Planck length. So our conclusion is that with this value of ϵ_P , the quantum coupling constant of gravity becomes dimensionless, and we have $\kappa_0 \approx G\sqrt{\epsilon_P} \approx 1$, so κ_0 is around unity as the coupling constants of the other interactions.

1.12 Tautological signature of unification

What will appear constantly in [56] and [57], in which we deal with unification, is that unification has a signature, a mark, which appears every time it appears, and this mark is tautology. Here, we just give an example of it. Why is gravitation so tiny? In fact we should ask why G is so tiny, and from (1.3) with κ_0 around unity, the question is to know why ϵ_P is so great. Now considering the masses of the particles as fixed quantities, ϵ_P is so great because the radiuses of the particles are so small. Why now the radiuses of the particles are so small? They appear to us so small because we are so big compared to them. And we are so big compared to them simply because we are made of them. Particles are by definition the smallest pieces of matter, and unification makes appear that they are effectively very small.

1.13 Dimensionless cosmology

If the second part of this work [56] will be devoted to unification and the calculation of the parameters of the standard model, in the third part [57] we enter into dimensionless

physics where all units are eliminated. We prove that if the quantum equation of gravity permits to solve the cosmological problems left over by general relativity, as it is proved in the present article, dimensionless cosmology also permits not to lose the beautiful achievements of the standard cosmological model which are the calculation of the abundances of the elements and the explanation of the cosmic background radiation. In fact we can even improve the standard calculation in two ways : first we can improve the computed value of the mass fraction $Y \approx 0.28$ of helium and push it into the observed region $0.22 \leq Y \leq 0.23$ (Pagel and al, 1994 [47]). Second, we are able to describe a method which permits to relax the condition on the energy density of baryonic dark matter, and this way, we can relax the hypothesis of the existence of non baryonic dark matter, as well as we can relax the condition that there are no more than three families of particles.

2 The complete deduction of the equation

2.1 Introduction

We now apply our ideas to construct quantum gravity. Applying the minimal principle, we consider all observations of the cosmos as indications of the tensorial equation inside matter. Then, we explain how can be constructed this equation. We will see that considering the experimental data coming from the observation of our universe, and also the necessary conditions imposed by the fact that the quantization of gravity has to obey certain rules that renders it coherent with the quantization of the other interactions, is far enough to deduce the rules of tensorial quantum gravity itself.

2.2 Perturbative corrections

2.2.1 Conditions for quantum gravity

So we first know that quantum gravity should smooth out the initial singularity. But it has a second and a third conditions to fulfill, this because as any other interaction,

it has to follow some characteristic features of renormalization. The second condition is that its coupling constant has to be dimensionless. Furthermore we know that once renormalized, the coupling constant of any interaction depends on energy. All the same, the third condition on the gravitational constant of gravitation is that it should vary with energy in the quantum regime.

2.2.2 Dependence of the gravitational constant on energy

So quantum gravity has to fulfill the former three conditions. In the early times of the universe, the energy density grew to $+\infty$. We suspect that a gravitational constant, which is a sufficiently rapidly decreasing function of energy, and such that

$$\lim_{\epsilon \rightarrow +\infty} G(\epsilon) = 0 \tag{2.1}$$

smooths out the initial singularity. An explanation for this is the picture of a big crunch : if the universe shrinks to a point in finite time, we can suspect gravitation to be the cause of this singularity, because it is an attractive interaction. When the universe shrinks to a point, $\epsilon \rightarrow +\infty$, and the former equation (2.1) makes gravitation disappear. We thus expect the big crunch singularity to be smoothed out. By analogy, we expect in this case the big bang to be smoothed out too. The precise proof of this fact will have to wait until we possess the complete set of equations of quantum gravity. This fact will be proved rigorously at that time. Now, if we take $\hbar = c = 1$, in the equation

$$8\pi G(\epsilon) = \kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\epsilon}} \tag{2.2}$$

the new constant κ_0 is dimensionless, and our three conditions are likely to be satisfied. To keep the most general equation, we can eventually allow κ_0 to depend on ϵ too, to take into account further perturbative corrections, in this case we keep the notation $\kappa(\epsilon)$.

2.3 Nonperturbative quantum corrections

2.3.1 Conservation of energy

There is, due to the conservation of energy, a relation of some kind between perturbative and nonperturbative corrections in tensorial quantum gravity, as we can see as soon as we write down the equations of general relativity with a constant of gravitation depending on ϵ :

$$R_{ik} - \frac{1}{2}Rg_{ik} = \kappa(\epsilon)T_{ik} \quad (2.3)$$

The left hand side of this equation, the Einstein's tensor, fulfills the conservation of energy. Also does the energy-momentum tensor of matter T_{ik} , and we can apply the covariant derivative to this equation, and contract it with one index i , to prove that $\kappa(\epsilon)$ is independent of ϵ . So, if the gravitational coupling constant really depends non trivially on energy, the only possibility is that we missed a term somewhere. This term corresponds precisely to nonperturbative corrections.

2.3.2 Nonperturbative corrections

Looking at what happens in the case of the other interactions, we see that in these cases there is also nonperturbative terms called instantons which are obtained by adding a topological term to the lagrangian. So our entire equation should also contain a topological term to fulfill all the requirements of an equation describing the complete quantum regime of gravity. The only topological term in dimension four is the Gauss-Bonnet term. We emphasize here that we are not looking for truly gravitational instantons, but we only retain the idea that we need a topological term to insert into our equation. We call $\tilde{\Sigma}$ the Gauss-Bonnet term and define $\tilde{\Sigma} = 4\tilde{\tilde{\Sigma}}$ which will simplify the calculations :

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2 = 4\tilde{\tilde{\Sigma}} \quad (2.4)$$

where $R^{(4)} = R^{abcd}R_{abcd}$ and $R^{(2)} = R^{ab}R_{ab}$, R_{abcd} being the Riemann curvature tensor, R_{ab} the Ricci tensor, and R the scalar curvature. So, putting this term on the left on our

equation we obtain :

$$R_{ik} - \frac{1}{2}Rg_{ik} + \Lambda g_{ik} = \kappa(\epsilon)T_{ik} \quad (2.5)$$

with :

$$\Lambda = -\theta\tilde{\Sigma} + \Lambda_0 = -\frac{\theta\tilde{\Sigma}}{4} + \Lambda_0 \quad (2.6)$$

θ being constant. The constant Λ_0 is here inserted for completeness, to write down the most general equation. When resolving the equation, we will adopt one strategy. We will impose certain conditions on the solutions and choose Λ_0 to obtain them. This way Λ_0 will appear as a kind of constant of integration. This way we will be able to study and test the general features of our quantum equation. At the end, we will leave all these models and impose the better condition $\Lambda_0 = 0$. Now, applying the covariant derivative on equation (2.5) we see that $\kappa(\epsilon)$ has to depend on ϵ because $\tilde{\Sigma}$ is not constant. Exactly, the conservation of energy gives a precise relation between perturbative and nonperturbative quantum gravitational corrections.

2.3.3 A problem

We can now write a tensorial equation of quantum gravity inside matter, but a problem appears. If the gravitational constant depends on energy, why has this fact never been detected by experiments in the solar system or even on earth? This new form of the gravitational constant should change drastically even the approximate Newton's law. We emphasize that there are numerous solutions to this problem, and that these solutions all give very different physics. We have studied them at length in [58], and we have proved that an answer to this problem necessarily involves more fundamental principles about unification. We will not need this complete study in the present work. Here we only consider the case where the gravitational coupling is spatially constant in the whole universe, and varies in time proportionally to $1/\sqrt{\epsilon}$, where ϵ is the mean energy density in the universe, as in any other cosmological model with varying G . Here we precise that ϵ is a global parameter which is spatially constant, because it is the mean value over large parts of space of the local parameter ϵ . When, as we do here, we take for ϵ the global parameter, the problem of retrieving Newton's theory and general relativity from our

quantum equation of gravity disappears, our gravitational coupling is spatially constant, and varies very little with time. So our quantum equation is very near general relativity, and we do not need all the technicality of the intermediate model, of the different pictures for particles, and of the assertions on how far we can go using only general relativity in vacuum. When ϵ in the quantum equation is taken to be the local parameter, on the contrary, all this technicality is necessary, and as we proved in the introduction, from this point of view, unification looks completely different. We need then averaging procedures to go from the equation with local parameters to the equation with global parameters. We simply remark here that with the equation with local parameters, the gravitational coupling constant varies spatially. However, one conclusion made in [58] is that there is only very little room to allow in our theory spatial variations of the gravitational coupling constant which could explain the phenomenon of dark matter in the halos of galaxies by theoretical means.

2.4 Generalizing the equation

2.4.1 Interpretation of the coefficient in front of the Gauss-Bonnet term

We called θ the constant in front of $\tilde{\Sigma}$, and considerations of dimensions show that θ has dimension $[L]^2$, where $[L]$ is a length. This way $\sqrt{\theta}$ has the same dimension as $[L]$. We will see that after the resolution of (2.5), (2.6), with constant θ , we find the relation :

$$\theta = \frac{c}{2H} \quad (2.7)$$

which gives, using the present value of H , the value :

$$\sqrt{\theta} \cong 1250 Mpc \quad (2.8)$$

So what can be such a huge length in our universe and what is its physical interpretation? It appears that such a distance can only be the greatest distance possible, that is, up to a constant of the order of unity, the radius of the universe itself, but considered in another model or equation. Another equation, because in this case, the equation is changed, and

the constant term θ is replaced by the term $\theta_0 a^2$, where θ_0 is constant and a is the radius of the universe.

2.4.2 The generalized equation

To write down the most general equation, we suppose that $\theta(a)$ is a function of the radius a , eventually constant or proportional to a^2 itself, but a priori yet undetermined. The dependence of θ on a can also be indirect : θ can depend on a only because it depends on ϵ . Using dimensions, in this case θ should be proportional to $1/\sqrt{\epsilon}$ which just changes a little its behavior, compared to $\theta \sim a^2$. So we begin to notice the emergence of a entire new class of cosmological models, which are quite near from each other, with only subtle differences. All these models will prove themselves to display the same general features, which could be called quantum features, and the differences will be numerous enough to make easy the task to find between all these possibilities, one or several models which fit with all observed data. Mathematically and for the time being, we still can write our θ as a function $\theta(a)$ and we finally write our most general equation :

$$R_{ik} - \frac{1}{2}Rg_{ik} - \theta(a)\tilde{\Sigma}g_{ik} = \kappa(\epsilon)T_{ik} \quad (2.9)$$

and :

$$\Lambda = -\theta(a)\tilde{\Sigma} + \Lambda_0 = -\theta(a)\frac{\tilde{\Sigma}}{4} + \Lambda_0 \quad (2.10)$$

2.5 Restoring h and c in the quantum gravitational coupling

Using (2.9), and since we also know that $T_0^0 = \epsilon$, we see that :

$$\left(R_0^0 - \frac{1}{2}R \right) - \theta\tilde{\Sigma} + \Lambda_0 = \kappa(\epsilon)\epsilon \quad (2.11)$$

We have

$$R_0^0 - \frac{1}{2}R \sim [L]^{-2}$$

where the last symbol means an equality of dimensions, and where $[L]$ is a length. Of course we have $[L] \sim c[T]$, where c is the speed of light and $[T]$ is a time. Now $\epsilon[L]^3$ is an

energy $[E]$, and :

$$\epsilon[L]^3 \sim [E] \sim \frac{\hbar}{[T]} \sim \frac{\hbar c}{[L]}$$

so

$$\epsilon \sim \frac{\hbar c}{[L]^4}$$

At the same time, the left hand side of equation (2.11) has dimension $[L]^{-2}$, and this is why the whole term on the right hand side should be proportional to $\sqrt{\epsilon}/\sqrt{\hbar c}$ in order to give us the dimension :

$$\frac{\sqrt{\epsilon}}{\sqrt{\hbar c}} \sim [L]^{-2}$$

We see that the coupling $\kappa(\epsilon)$ take the form :

$$\kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\hbar c}\sqrt{\epsilon}} \tag{2.12}$$

with κ_0 a dimensionless real number. We will still note this constant κ_0 when considering $\hbar = c = 1$.

Part II

The basic set of equations in tensorial quantum gravity

3 The quantum equations in cosmology

3.1 Introduction

We now take the quantum equation of gravity in order to apply it to the Robertson-Walker metric, and find this way the predictions of quantum gravity concerning cosmology. In

this part, we make all necessary computations to write down the complete set of quantum equations relating the cosmological parameters, and to prove that the quantum equation implies by itself the conservation of entropy. We recall that the quantum equation of gravity takes the form :

$$R_{ik} - \frac{1}{2}Rg_{ik} + \Lambda g_{ik} = \kappa(\epsilon)T_{ik} \quad (3.1)$$

with :

$$\Lambda = -\theta\tilde{\Sigma} + \Lambda_0 \quad (3.2)$$

θ can be constant or can depend of a , Λ_0 is left here for the moment to test solutions, even if we will take it equal to zero at the end. $\tilde{\Sigma}$ is proportional to the Gauss-Bonnet term, and equal to :

$$\tilde{\Sigma} = \frac{1}{4}(R^{(4)} - 4R^{(2)} + R^2) \quad (3.3)$$

where $R^{(4)} = R^{abcd}R_{abcd}$ and $R^{(2)} = R^{ab}R_{ab}$, R_{abcd} being the Riemann curvature tensor, R_{ab} the Ricci tensor, and R the scalar curvature.

3.2 The Robertson-Walker metric, notations

3.2.1 The hypothesis of isotropic and homogeneous space

We first stick to the closed model, we shall see in section 3.5 how can be deduced the equations of the open model from the closed one. As in any other cosmological model, we make the hypothesis of isotropic and homogeneous space. We know that this hypothesis implies that all our variables are averaged ones over large parts of space. We use coordinates such that free falling matter is at rest and t is the physical time given by physical free falling clocks. Our usual unknowns are the radius of the universe $a(t)$, the pressure $p(t)$ and the energy density $\epsilon(t)$. We let $\kappa(\epsilon)$ undetermined to test different behaviors of our solutions. We now write down all results found in Landau, [33], paragraphs 112 and 113, that will be of use in our present computation. We use the value of ds^2 coming from the Robertson-Walker metric :

$$ds^2 = c^2 dt^2 - a^2(t)(d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (3.4)$$

where r, θ, ϕ are the variables of the spherical coordinates in three dimensions and where $r = a(t) \sin \chi$, χ varying from 0 to π . Further, we can replace the time variable t , by the dimensionless variable η , defined by :

$$cdt = ad\eta \quad (3.5)$$

Here we take the convention that an expression as a' means an η -derivative and an expression as \dot{a} means a t -derivative. We then obtain :

$$ds^2 = a^2(\eta)(d\eta^2 - d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)). \quad (3.6)$$

We write our equations with variables x^0, x^1, x^2, x^3 being η, χ, ϕ, θ . We have from the previous equation :

$$g_{00} = a^2; g_{11} = -a^2; g_{22} = -a^2 \sin^2 \chi; g_{33} = -a^2 \sin^2 \chi \sin^2 \theta \quad (3.7)$$

all non diagonal terms of g_{ik} vanishing.

3.3 Values of tensors

We choose to use greek indices to denote space indices varying from 1 to 3, and latin indices to go from 0 to 3. We compute in Part VII the components of the Ricci tensor. These can also be found in [33] :

$$R_{\alpha 0} = 0 \quad (3.8)$$

and :

$$R_0^0 = \frac{3(a'^2 - aa'')}{a^4} = b \quad (3.9)$$

which defines b , as well as :

$$R_{\beta\delta} = -\frac{1}{a^4} (2a^2 + a'^2 + aa'') g_{\beta\delta} = cg_{\beta\delta} \quad (3.10)$$

which defines c . Now the scalar curvature :

$$R = b + 3c = -\frac{6}{a^3}(a + a'') = d \quad (3.11)$$

which defines d . In Parts VI and VII we define the tensor :

$$\Sigma_{ik} = \tilde{\Sigma}_{ik} - \frac{1}{4}\tilde{\Sigma}g_{ik} \quad (3.12)$$

where :

$$\tilde{\Sigma}_{ik} = R_i^{abc}R_{kabc} - 2R_{iakb}R^{ab} - 2R_{ia}R_k^a + R_{ik}R \quad (3.13)$$

and where $\tilde{\Sigma}$ is the trace of $\tilde{\Sigma}_{ik}$. We have the relations

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2$$

and

$$\tilde{\Sigma} = \frac{1}{4}\tilde{\Sigma}$$

In this Part VII we compute :

$$\tilde{\Sigma}_{\alpha 0} = 0 \quad (3.14)$$

we also compute :

$$\tilde{\Sigma}_{00} = \frac{b}{3}(3c - b)g_{00} \quad (3.15)$$

and :

$$\tilde{\Sigma}_{\alpha\beta} = \frac{b}{3}(3c - b)g_{\alpha\beta} \quad (3.16)$$

Finally, we also prove in Part VII that $\tilde{\Sigma}_{ik}$ is diagonal and that Σ_{ik} , being at the same time diagonal and of vanishing trace, verifies $\Sigma_{ik} = 0$ in the case of the homogeneous and isotropic model. This was known from topological arguments, and the arguments we give in Part VI. This situation is especially interesting because it acts as a theoretical check of all the calculations, in Part VII, which lead to (3.14), (3.15) and (3.16). We straightforwardly deduce :

$$\tilde{\Sigma} = 4\frac{b}{3}(3c - b) \quad (3.17)$$

and

$$\tilde{\Sigma} = \frac{b}{3}(3c - b) \quad (3.18)$$

3.4 The equations

3.4.1 The equations for the Gauss-Bonnet term

We start from (3.1) :

$$R_{ik} - \frac{1}{2}g_{ik} + \Lambda g_{ik} = \kappa(\epsilon)T_{ik}$$

and make operate a covariant derivative, as as in the conservation of energy :

$$\nabla^i \left(R_{ik} - \frac{1}{2}g_{ik} \right) + \partial_k \Lambda = \partial_i \kappa(\epsilon) T^i_k + \kappa(\epsilon) \nabla^i T_{ik} \quad (3.19)$$

and we are left with :

$$\partial_k \Lambda = \partial_i \kappa(\epsilon) T^i_k \quad (3.20)$$

but we have $\partial_\alpha \epsilon = 0$ so $\partial_\alpha \kappa(\epsilon) = \kappa'(\epsilon) \partial_\alpha \epsilon = 0$. As well T^i_k is diagonal, and $T^0_0 = \epsilon$. So we find :

$$\dot{\Lambda} = \kappa'(\epsilon) \epsilon \dot{\epsilon} \quad (3.21)$$

and :

$$\partial_\alpha \Lambda = \partial_\alpha \kappa = 0 \quad (3.22)$$

Reading the value of $\tilde{\Sigma}_{ik}$ in (3.15) and in (3.16), we can write :

$$\tilde{\Sigma}_k^i = \frac{b}{3}(3c - b)\delta_k^i \quad (3.23)$$

Now using (3.2) and (3.18) we have :

$$\Lambda = \theta \frac{b}{3}(b - 3c) + \Lambda_0 \quad (3.24)$$

3.4.2 Computation of b and c

Using the usual $\hbar = c = 1$, from (3.5) it appears that $a' = a\dot{a}$ and also that $a'' = a^2\ddot{a} + a\dot{a}^2$.

Using (3.9), it yields :

$$b = \frac{-3\ddot{a}}{a} \quad (3.25)$$

It is also straightforward to compute c : from (3.10), using the former relations for \dot{a} and \ddot{a} , we find :

$$R_\alpha^\alpha = c = -\frac{1}{a^4} (2a^2 + a'^2 + aa'') = -\frac{1}{a^4} (2a^2 + 2a^2\dot{a}^2 + a^3\ddot{a})$$

and we have :

$$c = - \left(\frac{2}{a^2} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\ddot{a}}{a} \right) \right) \quad (3.26)$$

3.4.3 The equations of the quantum theory

We also have another equation coming from the conservation of entropy in the universe, which is computed in [33], and takes the form :

$$\dot{\epsilon} = -3(\epsilon + p) \frac{\dot{a}}{a} \quad (3.27)$$

As far our equations are concerned, the first equation of movement is :

$$\left(R_0^0 - \frac{1}{2}R \right) + \Lambda = \kappa(\epsilon)T_0^0 = \kappa(\epsilon)\epsilon \quad (3.28)$$

with :

$$R_0^0 - \frac{1}{2}R = b - \frac{1}{2}(b + 3c) = \frac{b - 3c}{2}$$

so we finally obtain :

$$\frac{1}{2}(b - 3c) + \Lambda = \kappa(\epsilon)\epsilon \quad (3.29)$$

We have as well a second equation of movement which is :

$$\left(R_\alpha^\alpha - \frac{1}{2}R \right) + \Lambda = \kappa(\epsilon)T_\alpha^\alpha = -\kappa(\epsilon)p \quad (3.30)$$

with :

$$R_\alpha^\alpha - \frac{1}{2}R = c - \frac{1}{2}(b + 3c) = -\frac{b + c}{2}$$

so we finally get :

$$\frac{b + c}{2} - \Lambda = \kappa(\epsilon)p \quad (3.31)$$

3.5 From the closed model to the open model

To deduce the equations for the open model from the closed one, we need to apply the rules given in Landau, [33], paragraph 113. These rules state that to go from closed to open, we have to replace η, χ, a by $i\eta, i\chi, ia$, and since we also have $cdt = ad\eta$, these rules

also imply to replace t by $-t$, which in particular must be done in time derivatives. In other words each time derivative must be affected by a extra minus sign. Looking at the values of b and c we have just established, it is clear that b remains unchanged :

$$b = \frac{-3\ddot{a}}{a} \quad (3.32)$$

Indeed \ddot{a} is multiplied by i in the open model, because we have one i for a and two minus signs for each of the two time derivatives, which finally cancel. a in the denominator is multiplied by i making b remaining unchanged. In c , we see that a term like a^2 is multiplied by a minus sign, because a was multiplied by i , also \dot{a}/a is multiplied by a minus sign, and its square remains unchanged. We note $K = +1$ for the closed case and $K = -1$ for the open case and we deduce straightforwardly that in both cases :

$$c = - \left(\frac{2K}{a^2} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\ddot{a}}{a} \right) \right) \quad (3.33)$$

4 Conservation of entropy

4.1 Dependence of the three equations

We know that in the case of constant $\kappa(\epsilon) = 8\pi G$, the equations of movement imply automatically the conservation of entropy. Now we would like to prove this fact in our present case with varying $\kappa(\epsilon)$. For this, we have to prove that the three equations (3.27), (3.29) and (3.31) are dependent. Adding (3.29) and (3.31) we obtain :

$$b - c = \kappa(\epsilon)(p + \epsilon) \quad (4.1)$$

and we can multiply (3.29) by 2 and derive :

$$\dot{b} - 3\dot{c} + 2\dot{\Lambda} = 2\kappa'(\epsilon)\dot{\epsilon}\epsilon + 2\kappa(\epsilon)\dot{\epsilon} \quad (4.2)$$

and we know that (3.21) is the contraction of the covariant derivative of the equation of motion, so that it is available as the law of conservation of energy, besides (3.29) and (3.31). Thus, we can use it in our present computation and find :

$$\dot{b} - 3\dot{c} = 2\kappa(\epsilon)\dot{\epsilon} \quad (4.3)$$

Here we see that, in our calculations, we took the derivation of one equation, which is (3.29). So the system we obtain now is only equivalent to the first one up to a constant of integration, which of course involves Λ_0 . As a check, we see that Λ does not appear anymore in the last three equations, it has been eliminated by combination or derivation. We then are left to prove that (4.1) and (4.3) produce by themselves (3.27). First sticking to the proof that these three equations are dependent, we shall prove later that effectively the first two imply the third. The small difference between these two statements is that when three equations are dependent, two of them imply the third, but we do not necessarily know which they are. To prove that the three equations are dependent, we know that we have these three equations for three unknown functions ϵ, p, a and we can use two of these equations to eliminate ϵ and p . We thus are left to prove that the third is an identity on a . We have

$$3\kappa(\epsilon)(\epsilon + p)\frac{\dot{a}}{a} = -\kappa(\epsilon)\dot{\epsilon} = 3(b - c)\frac{\dot{a}}{a} \quad (4.4)$$

the first equality comes from (3.27) and the second from (4.1), and we also have :

$$\frac{\dot{b} - 3\dot{c}}{2} = \kappa(\epsilon)\dot{\epsilon} = -3(b - c)\frac{\dot{a}}{a} \quad (4.5)$$

using first (4.3) and second (4.4). So we are left with :

$$\dot{b} - 3\dot{c} = 6(c - b)\frac{\dot{a}}{a} \quad (4.6)$$

Now we prove that (4.6) is trivial and the proof is finished. From (3.32) and (3.33) we have :

$$b - 3c = -\frac{3\ddot{a}}{a} + 3\left(\frac{2K}{a^2} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right) = 6\left(\frac{K + \dot{a}^2}{a^2}\right) \quad (4.7)$$

and

$$c - b = -2\left(\frac{K}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right) \quad (4.8)$$

If we note

$$\beta = \frac{b - 3c}{6} = \left(\frac{K + \dot{a}^2}{a^2}\right) \quad (4.9)$$

(4.8) and (4.9) show us that (4.6) is equivalent to :

$$\dot{\beta} = (c - b)\frac{\dot{a}}{a} = -2\left(\frac{K + \dot{a}^2 - a\ddot{a}}{a^2}\right)\frac{\dot{a}}{a}$$

or to :

$$a\dot{\beta} = -2 \left(\beta - \frac{\ddot{a}}{a} \right) \dot{a}$$

so finally equivalent to :

$$a\dot{\beta} + 2\beta\dot{a} = 2\frac{\dot{a}\ddot{a}}{a}$$

or to :

$$a^2\dot{\beta} + 2a\dot{a}\beta = 2\dot{a}\ddot{a}$$

or to :

$$\widehat{(\beta a^2)} = 2\dot{a}\ddot{a}$$

Now

$$\beta a^2 = \left(\frac{K + \dot{a}^2}{a^2} \right) a^2 = K + \dot{a}^2$$

and (4.6) is proved to be trivial.

4.2 Formal proof of the conservation of entropy from the quantum equation

We now give the formal proof that the two quantum equations effectively lead to the conservation of entropy. We follow the former proof : When we look at what has been needed to prove (4.6), we see that only (3.32) and (3.33) were required, two equations which are pure identities. So (4.6) is a pure identity following from the definitions of b and c . We suppose the two equations of movement (3.29) and (3.31), we suppose also the third equation obtained from the conservation of energy, that is we suppose (3.21). The first two yield directly (4.1) and with the help of the third we establish (4.3) as it has been done in the former section. In (4.5), they are three expressions. (4.3) gives the equality between the first two, and (4.6) the equality between the first and the third. So (4.5) is established. In (4.4) they are three expressions, (4.1) gives the equality between the first and the third, and (4.5) gives the equality between the second and the third. So (4.5) is completely established, and the equality of the first two expressions of this formula is exactly (3.27), that is to say the conservation of entropy.

4.3 Conservation of energy of matter in the quantum context

The condition (3.21) is the equation obtained after taking the covariant derivative of our quantum equation. It comes from the quantum equation, the conservation of energy of Einstein's tensor, and the conservation of energy of the stress-energy tensor of the matter fields. In general relativity, with or without the cosmological constant, (3.21) is automatically verified because in this case we have

$$\dot{\Lambda} = \kappa'(\epsilon) = 0 \quad (4.10)$$

So in general relativity, the conservation of energy of the matter fields is in fact put by hand in the equation by imposing that the cosmological constant necessitates to be constant. This condition in the quantum context is replaced by condition (3.21) which was used to prove the conservation of entropy. In fact, in the tensorial context, the conservations of entropy and of energy of the matter fields are two equivalent conditions. This property could not be stated in the context of general relativity, because since the conservation of energy was automatic, the conservation of entropy appeared only automatically verified, but not necessarily linked with the conservation of energy.

4.4 Equivalence between the conservations of energy and entropy

We suppose our two equations of movement, (3.29) and (3.31), plus the conservation of entropy (3.27). The derivation of (3.29) implies directly (4.2), which proves that the conservation of energy (3.21) is now equivalent to (4.3). Still, (4.6) is an identity, always available. (4.5) possesses three expressions, (4.6) implies the equality between the first and the third, the conservation of entropy and (4.1) prove the equality between the the second and the third, and (4.1) is implied by our two equations of movement. So these last two plus the conservation of entropy imply (4.5) completely. The equality of the first two expressions in (4.5) is exactly (4.3). So we have established the fact that if we suppose the two equations of movement, the conservation of entropy and the conservation of energy of the matter fields are equivalent.

5 Conclusion : a basic system of equations

We conclude this calculation by writing down a set of equations equivalent to the whole set of the quantum equations of gravity. We know that we first have the cosmological constant term, which is no more constant,

$$\Lambda = \theta \frac{b}{3}(b - 3c) + \Lambda_0 \quad (5.1)$$

We put now this value of Λ in our equation (3.21):

$$\dot{\Lambda} = \dot{\theta} \frac{b}{3}(b - 3c) + \theta \frac{\dot{b}}{3}(b - 3c) + \theta \frac{b}{3}(\widehat{b - 3c}) = \kappa'(\epsilon)\epsilon\dot{\epsilon} \quad (5.2)$$

Equation (5.2) is the law of conservation of energy of the matter fields, to be imposed on our quantum equation itself. We write now equation (4.3) :

$$\dot{b} - 3\dot{c} = 2\kappa(\epsilon)\dot{\epsilon} \quad (5.3)$$

(5.2) and (5.3) give (4.2), which is the derivation of one of the two equations of movement, namely (3.29). So equations (5.2) and (5.3) are equivalent to the conservation of energy and to the equation of movement (3.29) concerning only energy, up to a constant, since (4.2) is only the time derivative of (3.29). If we need to determine this constant, which happens to be Λ_0 , we need to apply (3.29) itself, which is :

$$\frac{1}{2}(b - 3c) + \Lambda = \kappa(\epsilon)\epsilon \quad (5.4)$$

Finally, there is another equation of movement, to determine p , which is (3.31) :

$$\frac{b + c}{2} - \Lambda = \kappa(\epsilon)p \quad (5.5)$$

Part III

The first equation of quantum gravity

6 Introduction : smoothing out the initial singularity

We call : first equation quantum gravity, the equation when the parameter θ is constant, and in this first part of the computation, we thus suppose θ constant. For the time being, we leave Λ_0 undetermined, as a constant of integration, and we will see that one value of this constant gives us, in the case of the early universe, classical inflation, defined as the exponential growth of a , and at the same time, the standard properties of all cosmological parameters, except a of course. We recall that at the end, in Parts IV and V, θ will be varying and Λ_0 will be put equal to zero. In this part, we stick to the case

$$\kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\epsilon}} \quad (6.1)$$

where κ_0 is strictly constant, and where we have supposed $\hbar = c = 1$. More precisely, we are going to prove that the quantum equation, for θ constant and strictly positive, implies that the whole set of cosmological parameters is the same as in the standard cosmological model applied to the early universe, except a which has now an exponential growth. This means that the quantum equation in this case gives simultaneously the relations $a(t) \sim e^{\chi t}$, $p > 0$, $p = \epsilon/3$, and $\epsilon \sim 1/a^4$. So we retrieve all features of the standard early universe except that we have smoothed out the initial singularity. Another feature of the quantum equation is that it implies by itself the $p = \epsilon/3$ relation which is not simply put by hand anymore, but is really a non trivial consequence of the quantum theory. We emphasize this point because it means the following conclusion : *one sole quantum equation of gravity governs at the same time the cosmological parameters of the universe, exactly as did the standard cosmological model, and the*

structure of fundamental particles, giving the right relation between p and ϵ , whereas, furthermore, it smoothes out the initial singularity with no further hypothesis.

7 Integration of the equations

We first stick to the computation of ϵ , and leave for the time being the equation for p . We are left with two equations which are (5.2) and (5.3). We use :

$$\kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\epsilon}}$$

with constant κ_0 . We find, from (5.2), since θ is constant :

$$\theta \frac{\dot{b}}{3}(b-3c) + \theta \frac{b}{3}(\widehat{b-3c}) = -\kappa_0 \frac{\dot{\epsilon}}{2\sqrt{\epsilon}} \quad (7.1)$$

and from (5.3) :

$$(\widehat{b-3c}) = 2\kappa_0 \frac{\dot{\epsilon}}{\sqrt{\epsilon}} \quad (7.2)$$

We combine these equations and find :

$$\theta \frac{\dot{b}}{3}(b-3c) + \theta \frac{b}{3}(\widehat{b-3c}) = -\frac{(\widehat{b-3c})}{4} \quad (7.3)$$

which, for $\theta \neq 0$, gives :

$$\theta \frac{\dot{b}}{3}(b-3c) + \theta \left(\frac{b}{3} + \frac{1}{4\theta} \right) (\widehat{b-3c}) = 0$$

or :

$$\theta \left(\frac{b}{3} + \frac{1}{4\theta} \right) (b-3c) + \theta \left(\frac{b}{3} + \frac{1}{4\theta} \right) (\widehat{b-3c}) = 0 \quad (7.4)$$

and we finally obtain :

$$\left(\frac{b}{3} + \frac{1}{4\theta} \right) (b-3c) = C \quad (7.5)$$

where C can be chosen at will since any C gives (7.4). In fact this is only true because we have let Λ_0 unspecified as a constant of integration, which has permitted us to use the

derivative (5.2) of one of our equations of motion, instead of the equation (5.4) itself. We still have obtained a system equivalent to the quantum equation, up to Λ_0 of course. Now it is clear that two different choices of C should give us two different choices of Λ_0 . Of course the simplest case is $C = 0$. In the next two sections, we always adopt the choice $C = 0$.

8 The quantum features of the closed early universe, positive case

8.1 Introduction

We call positive case the case in which the Gauss-Bonnet parameter θ is positive. We now make this special hypothesis on our equation, which is that $\theta > 0$. This hypothesis rules out the open model as showed below. This condition $\theta > 0$ is necessary if we want our quantum equation to display the smoothing out of the initial singularity. As far as the particular case of constant θ is concerned, we shall prove in the next section that the condition $\theta < 0$ is compatible with both the closed and open models, but keeps the initial singularity.

8.2 Inflation

From $C = 0$ in (7.5), we deduce :

$$\frac{b}{3} = -\frac{1}{4\theta} \tag{8.1}$$

because from (4.7) :

$$(b - 3c) = 6 \left(\frac{K + \dot{a}^2}{a^2} \right) > 0$$

Indeed, the last expression is strictly positive, and thus nonzero, because we shall prove in the following, using (8.9), that we are necessarily in the closed model, and thus we have

$K = +1$. Furthermore, since

$$b = -\frac{3\ddot{a}}{a}$$

we find :

$$\frac{\ddot{a}}{a} = +\frac{1}{4\theta} \quad (8.2)$$

With $\theta > 0$, we finally obtain the inflation solution :

$$a(t) = a_0 e^{\chi t} \quad (8.3)$$

8.3 Value of the Gauss-Bonnet parameter

We call θ the Gauss-Bonnet parameter. Now, from the former solution, we get easily

$$\frac{\ddot{a}}{a} = \chi^2$$

and find the value of χ in our solution :

$$\chi = \frac{1}{2\sqrt{\theta}} \quad (8.4)$$

To reinsert in this equation the constants \hbar and c we use that $\theta \sim [L]^2$ while $\chi \sim [T]^{-1} \sim c[L]^{-1}$, so

$$\chi = \frac{c}{2\sqrt{\theta}} \quad (8.5)$$

8.4 The values of c , $c-b$ and $b-3c$

Now we compute all our functions, to determine, first the behavior of ϵ , second the value of p . We also compute the value of θ and verify directly, as a check, that all our quantum equations are satisfied by our solutions. ϵ and p are given by (3.27) and (4.1) :

$$\dot{\epsilon} = -3(\epsilon + p)\frac{\dot{a}}{a}$$

and

$$b - c = \kappa(\epsilon)(p + \epsilon)$$

but as we know that

$$\frac{\dot{a}}{a} = \chi$$

we obtain :

$$\dot{\epsilon} = -3\chi(\epsilon + p) \quad (8.6)$$

We also have :

$$b = -3\frac{\ddot{a}}{a} = -3\chi^2 \quad (8.7)$$

and

$$c = -\left(\frac{2K}{a^2} + 2\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)\right) = -\left(\frac{2K}{a^2} + 3\chi^2\right) \quad (8.8)$$

So

$$c - b = -\frac{2K}{a^2} = -\kappa(\epsilon)(p + \epsilon) \quad (8.9)$$

and

$$b - 3c = 6\left(\frac{K}{a^2} + \chi^2\right) \quad (8.10)$$

We recall that for the closed model we have $K = +1$, whereas the open model, which gives $K = -1$, is ruled out by the relation (8.9). Indeed $K = -1$ corresponds to the wrong sign in this equation. We emphasize the fact that what is only ruled out in this case is the combination : open model plus constant θ . This matter could as well be interpreted as a clue that θ is varying. We shall see just below that this hypothesis is confirmed by other arguments. Nevertheless, we will see again, studying the abundance of the elements, that the open model doesn't fit well with the quantum equation. We stick until the end of this section to the closed model. To put the value of Λ_0 back in the equation, we need to find this constant in terms of θ or κ_0 , but not χ , which is a variable which belongs to the set of solutions.

8.5 The value of the energy density

We combine (8.6) and (8.9) to find :

$$\frac{\dot{\epsilon}}{3\chi} = -(\epsilon + p) = -\frac{2}{\kappa(\epsilon)a^2} = -\frac{2\sqrt{\epsilon}}{\kappa_0 a^2}$$

so :

$$\frac{\dot{\epsilon}}{2\sqrt{\epsilon}} = \widehat{(\sqrt{\epsilon})} = -\frac{3\chi}{\kappa_0 a_0^2} e^{-2\chi t}$$

we integrate this equation and find :

$$\sqrt{\epsilon} = \frac{3}{2\kappa_0 a_0^2} e^{-2\chi t} = \frac{3}{2\kappa_0 a^2} \quad (8.11)$$

where we have chosen the simplest constant of integration without investigating all solutions. We see then that we obtain the behavior for ϵ which has also been found in the standard cosmological model. Indeed, we have

$$\epsilon a^4 = \frac{9}{4\kappa_0^2} = Cte$$

Reestablishing the values of \hbar and c :

$$\epsilon a^4 = \frac{9\hbar c}{4\kappa_0^2} = Cte \quad (8.12)$$

8.6 The value of the pressure from the quantum equation

Now we know that the behavior of a is $a = a_0 e^{\chi t}$. This gives, using (8.12), the behavior of ϵ : $\epsilon = \epsilon_0 e^{-4\chi t}$. We thus have :

$$\dot{\epsilon} = -4\chi\epsilon \quad (8.13)$$

and from (8.6) we see that :

$$\epsilon + p = -\frac{\dot{\epsilon}}{3\chi} = \frac{4}{3}\epsilon$$

We conclude what we wanted :

$$p = \frac{\epsilon}{3} \quad (8.14)$$

As claimed, this equation has not been put by hand but is the a consequence of the quantum equation of gravity itself.

9 The quantum features of the early universe in the negative case

We suppose in this section that $\theta < 0$, and θ still constant. We prove that in the open case, the quantum equation still leads to the behavior of ϵ as in the standard cosmological model, and furthermore that it still contains the information on the structure of matter. In other words, we prove in this open case that the quantum equation leads to the relation $p = \epsilon/3$. We then look for the expression of Λ_0 that renders the quantum equation possible in the open model and find that it has to be proportional to H .

9.1 Behavior of the radius of the universe

Again, we start from (7.5) and obtain :

$$\frac{b}{3} = -\frac{1}{4\theta} \quad (9.1)$$

and, since

$$b = -\frac{3\ddot{a}}{a}$$

we find :

$$\frac{\ddot{a}}{a} = +\frac{1}{4\theta} \quad (9.2)$$

With $\theta < 0$, supposing that $a = 0$ for $t = 0$, we obtain the solution :

$$a(t) = a_0 \sin \chi t \quad (9.3)$$

9.2 Value of Gauss-Bonnet parameter

Now from this solution, we get easily

$$\frac{\ddot{a}}{a} = -\chi^2$$

and find the value of χ in our solution :

$$\chi = \frac{1}{2\sqrt{-\theta}} \quad (9.4)$$

To reinsert in this equation the constants \hbar and c we use that $(-\theta) \sim [L]^2$ while $\chi \sim [T]^{-1} \sim c[L]^{-1}$, so

$$\chi = \frac{c}{2\sqrt{-\theta}} \quad (9.5)$$

9.3 The values of c , $c-b$ and $b-3c$

As in the former section, we compute all our functions, to determine, first the value of p , then the behavior of ϵ . Then we will compute the value of θ and verify directly, as a check, that all our quantum equations are satisfied by our solutions. ϵ and p are given by (3.27) and (4.1) :

$$\dot{\epsilon} = -3(\epsilon + p)\frac{\dot{a}}{a}$$

and

$$b - c = \kappa(\epsilon)(p + \epsilon)$$

but as we know that

$$H = \frac{\dot{a}}{a} = \chi \frac{\cos \chi t}{\sin \chi t} = \chi \cot \chi t \quad (9.6)$$

we have :

$$\dot{\epsilon} = -3\chi(\epsilon + p) \cot \chi t \quad (9.7)$$

We also have :

$$b = -3\frac{\ddot{a}}{a} = 3\chi^2 \quad (9.8)$$

and

$$c = -\left(\frac{2K}{a^2} + 2\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)\right) = -\left(\frac{2K}{a^2} + 2\chi^2 \cot^2 \chi t - \chi^2\right) \quad (9.9)$$

So

$$\begin{aligned} b - c &= \frac{2K}{a^2} + 2\chi^2 \cot^2 \chi t + 2\chi^2 = \frac{2K}{a^2} + \frac{2\chi^2}{\sin^2 \chi t} \\ &= \frac{2(K + (a_0\chi)^2)}{a^2} = \kappa(\epsilon)(p + \epsilon) \end{aligned} \quad (9.10)$$

We note $\mu = K + (a_0\chi)^2$ and find that

$$b - c = \frac{2\mu}{a^2} = \kappa(\epsilon)(p + \epsilon) \quad (9.11)$$

We also have :

$$b - 3c = 6 \left(\frac{K}{a^2} + \chi^2 \cot^2 \chi t \right) \quad (9.12)$$

9.4 The value of the energy density

We had :

$$\dot{\epsilon} = -3\chi(\epsilon + p) \cot \chi t = \frac{b - c}{\kappa(\epsilon)} (-3\chi) \cot \chi t$$

So we have :

$$\dot{\epsilon} = \frac{2\mu}{\kappa_0 a^2} (-3\chi) \sqrt{\epsilon} \cot \chi t$$

and :

$$\frac{\dot{\epsilon}}{\sqrt{\epsilon}} = 2 \frac{\dot{(\sqrt{\epsilon})}}{(\sqrt{\epsilon})} = \frac{3\mu}{\kappa_0 a_0^2} \frac{-2\chi \cos \chi t}{\sin^3 \chi t}$$

we integrate this equation and find:

$$\sqrt{\epsilon} = \frac{3\mu}{2\kappa_0 a_0^2} \frac{1}{\sin^2 \chi t} = \frac{3\mu}{2\kappa_0 a^2} \quad (9.13)$$

where we have chosen the simplest constant of integration without investigating all solutions. We see then that we obtain again the same standard behavior for ϵ :

$$\epsilon a^4 = \frac{9\mu^2}{4\kappa_0^2} = Cte$$

Reestablishing the values of \hbar and c :

$$\epsilon a^4 = \frac{9\mu^2 \hbar c}{4\kappa_0^2} = Cte \quad (9.14)$$

9.5 Consistency with both the open and closed models

From the former calculation, we see that the only condition that needs to be realized to make things possible is $\mu > 0$. In the closed model this condition is always verified,

whereas in the open model, since $K = -1$, it is verified provided we have $a_0\chi > 1$. We notice that the term $a_0\chi$ is the value of \dot{a} when $t = 0$ and $a = 0$. This condition has been smoothed compared to the situation in the standard cosmological model where $\dot{a} \rightarrow +\infty$ near the initial singularity. This is because the quantum equation naturally tends to smooth singularities. In the open model, with θ being constant and negative, it cannot do this job too properly, but it still improves the behavior of the cosmological parameters.

9.6 The pressure from the quantum equation

We retrieve the usual behavior of ϵ : $\epsilon = \epsilon_0 a^{-4}$ so we have :

$$\dot{\epsilon} = -4\epsilon \frac{\dot{a}}{a} \tag{9.15}$$

and from (3.27) we see that :

$$\epsilon + p = -\frac{\dot{\epsilon}a}{3\dot{a}} = \frac{4}{3}\epsilon$$

We conclude :

$$p = \frac{\epsilon}{3} \tag{9.16}$$

As claimed, this equation has not been put by hand but is the the result of the quantum equation of gravity itself.

10 Introduction to the flatness and cosmological constant problems in the context of quantum gravity

10.1 Quantum gravity implies that the universe had no beginning

The equation of quantum gravity, in the case of the early universe, and in the particular case in which θ is a positive constant, leads to an exponential growth for the cosmological

parameter a of the Robertson-Walker metric. This satisfies the principle which states that this smoothing out of the initial singularity should be the consequence of a fundamental law of nature. We recall that we stated this principle because this smoothing out was equivalent to the fact that the universe had no beginning. This absence of beginning was needed because such a beginning would have no former cause.

10.2 The flatness problem

We know that the quantum equation displays also characteristic features of the standard cosmological model, like the behavior of p and ϵ :

$$p = \frac{\epsilon}{3} \sim \frac{1}{a^4} \quad (10.1)$$

As a preliminary exercise, we can look at what gives us the equation of the standard cosmological model when the quantum behavior of a is put in it. General relativity gives the equation :

$$\frac{K}{a^2} + H^2 = \frac{8\pi G\epsilon}{3} \quad (10.2)$$

Concerning the present universe, we should take t quite large. So we look at the former equation when $t \rightarrow +\infty$. With the exponential growth of a coming from quantum gravity, we see that, in the last equation :

$$\frac{K}{a^2} = o(H^2) \quad (10.3)$$

We mean by this notation that two functions $f(t)$ and $g(t)$ verify $f(t) = o(g(t))$, if and only if

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{g(t)} = 0$$

We then obtain that the usual ratio Ω tends to 1 when $t \rightarrow +\infty$:

$$\Omega = \frac{8\pi G}{3H^2} \rightarrow 1 \quad (10.4)$$

This is an even better behavior of Ω than in the classical inflation model, since this time Ω tends smoothly to its observed value. We emphasize that this argument is only a guess since it uses the equation of general relativity which is not part of the quantum theory. We shall prove in this part that in fact the case of constant θ does solve the flatness problem, but with different arguments for the cases $\theta > 0$ and $\theta < 0$.

10.3 The cosmological constant problem

After this, we will ask ourselves if quantum gravity can solve the cosmological constant problem too. We will see that even in the case of positive constant θ , this problem already finds a solution. In the present part, in order to understand the general features of how these various cosmological problems can be solved by quantum gravity, we stick to the well known case of constant θ , case in which all values of all cosmological parameters can be computed exactly. We first compute the value of Λ_0 in this model, we then study the expansion, flatness and cosmological constant problems in this same context.

11 Value of the constant term

We call Λ_0 the constant term. In this section we prove that the values of the constant Λ_0 in both cases, when θ is a positive constant, and when θ is a negative constant, are the same, in the sense that this value is entirely determined by θ , and that in both cases, the formulas giving Λ_0 from θ are identical.

11.1 The positive case : value of the constant term

We recall that in this case, we are automatically in the closed model and that $K = +1$. We compute, using (5.1) and (8.7) :

$$\Lambda = \frac{\theta b}{3}(b - 3c) + \Lambda_0 = -\theta \frac{\ddot{a}}{a}(b - 3c) + \Lambda_0 = -\theta \chi^2(b - 3c) + \Lambda_0 \quad (11.1)$$

We know from (8.4) that $\theta \chi^2 = 1/4$ and we obtain, using (8.10) too, for $K = +1$:

$$\Lambda = \frac{(3c - b)}{4} + \Lambda_0 = -\frac{3}{2} \left(\frac{1}{a^2} + \chi^2 \right) + \Lambda_0 \quad (11.2)$$

We know from previous calculations, in particular from (3.29), that :

$$R_0^0 - \frac{1}{2}R = \frac{(b - 3c)}{2} = 3 \left(\frac{1}{a^2} + \chi^2 \right) \quad (11.3)$$

The first of our equations (3.28) was :

$$R_0^0 - \frac{1}{2}R + \Lambda = \kappa(\epsilon)\epsilon \quad (11.4)$$

so we find :

$$\frac{3}{2} \left(\frac{1}{a^2} + \chi^2 \right) + \Lambda_0 = \kappa_0 \sqrt{\epsilon} \quad (11.5)$$

and using the expression of ϵ found in (8.11), we obtain :

$$\Lambda_0 = -\frac{3}{2}\chi^2 \quad (11.6)$$

We find :

$$\Lambda_0 = -\frac{3}{2}\chi^2 = -\frac{3}{8\theta} \quad (11.7)$$

We also can write the value of Λ_0 in terms of Hubble's constant :

$$\Lambda_0 = -\frac{3}{2}\chi^2 = -\frac{3}{2}H^2 \quad (11.8)$$

which is of the order of H^2 , so tiny enough to be plausible. Furthermore we see that the quantum equation determines by itself the value of the cosmological constant term and gives exactly the value we expect, we mean of the order of H^2 . We shall see in the next part that indeed, our equation solves the cosmological problem. So finally, what conclusion can we draw about this constant Λ_0 ? In the case of constant θ and $\theta > 0$, we put in the equation the Λ_0 term, as a Λ term, which is always possible, since Λ_0 is constant. We can determine the value of Λ_0 , using a principle of pure logic : if ϵ is the mean energy density in the universe, the conservation of energy implies that $\epsilon \rightarrow 0$ when $a \rightarrow +\infty$. Comparing this condition to equation (11.5), it yields :

$$\Lambda_0 = -\frac{3}{2}\chi^2$$

Using the parameters of the equation itself, κ_0 and θ , Λ_0 has the value :

$$\Lambda_0 = -\frac{3}{8\theta}$$

Finally with this value, we find

$$\Lambda_0 = -\frac{3}{2}H^2$$

which has the right sign and the right order of magnitude to be a solution to the cosmological constant problem.

11.2 The negative case: value of the constant term

We still consider the case of constant θ , but now $\theta < 0$, and we compute, using (5.1), (9.2) and (9.4) :

$$\Lambda = \frac{\theta b}{3}(b - 3c) + \Lambda_0 = -\theta \frac{\ddot{a}}{a}(b - 3c) + \Lambda_0 = \theta \chi^2(b - 3c) + \Lambda_0 \quad (11.9)$$

We know that $\theta \chi^2 = -1/4$ and we obtain, using (9.12) :

$$\Lambda = \frac{(3c - b)}{4} + \Lambda_0 = -\frac{3}{2} \left(\frac{K}{a^2} + \chi^2 \cot^2 \chi t \right) + \Lambda_0 \quad (11.10)$$

We know, from (3.28) and (3.29), that :

$$R_0^0 - \frac{1}{2}R = \frac{(b - 3c)}{2} = 3 \left(\frac{K}{a^2} + \chi^2 \cot^2 \chi t \right) \quad (11.11)$$

Using (3.28) :

$$R_0^0 - \frac{1}{2}R + \Lambda = \kappa(\epsilon)\epsilon \quad (11.12)$$

we find :

$$\frac{3}{2} \left(\frac{K}{a^2} + \chi^2 \cot^2 \chi t \right) + \Lambda_0 = \kappa_0 \sqrt{\epsilon} \quad (11.13)$$

and using the expression of ϵ found in (9.13) :

$$\kappa_0 \sqrt{\epsilon} = \frac{3\mu}{2a^2}$$

we obtain :

$$\frac{K}{a^2} + \chi^2 \frac{\cos^2 \chi t}{\sin^2 \chi t} + \frac{2}{3}\Lambda_0 = \frac{\mu}{a^2} = \frac{K}{a^2} + \frac{(a_0 \chi)^2}{a^2} = \frac{K}{a^2} + \frac{\chi^2}{\sin^2 \chi t} \quad (11.14)$$

We conclude :

$$\Lambda_0 = \frac{3}{2}\chi^2 \quad (11.15)$$

for both models, closed and open. We find :

$$\Lambda_0 = \frac{3}{2}\chi^2 = -\frac{3}{8\theta} \quad (11.16)$$

which is the same formula than in the $\theta > 0$ model. We cannot write the value of Λ_0 in terms of Hubble's constant anymore, because now this one varies with time whereas Λ_0 is a true constant, obtained directly from the parameters of the equation.

12 The expansion problem

12.1 The problem

The expansion problem is explained in detail for example in Peacock 1999, 2005 [48], Chapter 11, section 11.1 : ”... Nevertheless it is the only level of explanation that classical cosmology offers : the universe expands now because it did so in the past. Although it is not usually included one might thus with justice add an ”expansion problem” as perhaps the most fundamental in the catalogue of classical cosmological problems. Certainly, early generations of cosmologists were convinced that some specific mechanism was required in order to explain how the universe was set in motion.”

12.2 The quantum behavior of the radius of the universe

In classical cosmology, the parameter a has such a behavior that $\ddot{a} < 0$. In these conditions, the deceleration parameter

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \quad (12.1)$$

is positive. In the seventies, one could read (Weinberg [67]) that the observed value of q is around 1/2, most certainly positive. This positive value of q , equivalent to a negative \ddot{a} , is at the origin of the expansion problem. An expanding universe means $\dot{a} > 0$, and the solution to the expansion problem is equivalent to the understanding of the condition $\dot{a} > 0$, with no further hypothesis on the initial conditions concerning \dot{a} . The condition $\ddot{a} < 0$ implies initial conditions where \dot{a} was even greater. In the quantum regime, where $q = -1$, the expansion problem is solved at once : with θ constant and positive we have $\ddot{a} > 0$, the positivity of \dot{a} is explained with no further hypothesis. Indeed, in the context of the exponential growth of a , the value of \dot{a} tends to zero when $t \rightarrow -\infty$, and this condition itself is linked to the absence of beginning we were looking for. So the problem of quantum gravity is not the expansion problem, which it solves easily, but the value of q . For this reason, we tried to improve this value by making θ vary. What we found is that we can make q tend to zero when $t \rightarrow +\infty$, but still with q negative. There is

no choice of varying θ that can make \ddot{a} negative. In other words, the only possibility for $\ddot{a} < 0$ is θ constant and negative. Finally, except for this particular case, the solution of the expansion problem is always provided by the quantum equation. On the other side, the measured value of q has been revised with time. It appears today that negative values of q are even more probable, and $q = -1$ is not ruled out anymore. Different experiments have made appear the fact that \ddot{a} could be positive. All these experiments of course play in favor of the quantum equation.

13 The cosmological constant problem

13.1 Comparison of theories : inertia

In order to study the cosmological constant problem, we compute the total value of the cosmological term Λ in the case of constant θ . This term comes from the topological Gauss-Bonnet term and from Λ_0 . We see that in our equation, this term does not come from any effective dark energy, but from the quantum corrections to classical gravity. The only unexplained term is Λ_0 but it does not matter since in the case of varying θ , we shall take $\Lambda_0 = 0$. In general relativity, to compute the total value of Ω , we see that the cosmological constant term is negative on the left hand side of the equation. We pass it on the other side, where it adds as a positive term to usual matter. This operation done, we have on the right hand side the total energy density of matter, counted with the supposed dark energy, and the equation makes it equal of course to the left hand side. The left hand side in this context is :

$$G_0^0 = R_0^0 - \frac{1}{2}R \quad (13.1)$$

We divide this expression by $3H^2$ to find Ω_{TOT} , which is Ω calculated with all kinds of matter together, even with dark energy, itself generated by Λ . We thus find the following formula for Ω_{TOT} :

$$\Omega_{TOT} = \frac{1}{3H^2} \left(R_0^0 - \frac{1}{2}R \right) \quad (13.2)$$

We emphasize that the last formula is the one which has to be used in order to compute the value of Ω_{TOT} in the context of the quantum equation of gravity. Indeed, the difference of the behaviors of general relativity and quantum gravity relatively to the parameter ϵ , makes difficult to compare the right hand sides of these two equations. This has a natural explanation, that we give just right now. First we cannot compute the value κ_0 so easily. We shall prove in [58] that this is a characteristic of all tensorial equations to possess a paradox in respect to their right hand side. Because they are in fact effective equations, that should be derived from unification, they still can be interpreted from different points of view, which changes at least the value we should take for κ_0 . These tensorial equations in fact all belong to theories which treat gravitation as an inertial interaction. Since Newtonian gravity is a limit case of general relativity, and since in general relativity, gravity is a pure effect of inertia, we deduce that in some sense Newtonian gravity is also purely inertial. Thus, in our observations of the sky, our experiments, when compared to general relativity, can only give us information about the properties of inertia of the celestial bodies, that is to say about the geodesics of space-time. All this information being locked in G_0^0 . Indeed, the equation of general relativity containing G_0^0 , the sole equation which is used to compute the masses, to compare general relativity to Newton's theory, and compute the relativistic gravitational coupling constant from Newton's G . The equation containing G_α^α is used only to compute the value of p . These arguments give the principle that the dimensionless term Ω_{TOT} should be defined in both general relativity and quantum gravity by the term controlling the equations of the geodesics in vacuum, so in both cases by G_0^0 , and we find

$$\Omega_{TOT} = \frac{1}{3H^2} G_0^0 \quad (13.3)$$

Using the equation of quantum gravity, we obtain :

$$3H^2 \Omega_{TOT} = G_0^0 = 2\kappa_0 \sqrt{\epsilon} \quad (13.4)$$

In the standard equation of general relativity, the cosmological constant is affected by a minus sign. Our equation, on the left hand side, contains the Λ term with a positive sign, so our quantum Ω_Λ , which is the dimensionless energy density, reads :

$$\Omega_\Lambda = -\frac{\Lambda}{3H^2} \quad (13.5)$$

We recall that within the realm of the standard cosmological model, Bennett and al. 2003, [2], experiments on the cosmic microwave background radiation imply the fairly precise relation :

$$\Omega_\Lambda = \frac{3}{4}\Omega_{TOT} \quad (13.6)$$

The quantum equation of gravity displays a natural energy density coming from Λ itself, composed of two elements, the Gauss-Bonnet term and Λ_0 . These two terms have different origins : the topological Gauss-Bonnet term is interpreted as a non perturbative quantum correction to classical gravity, Λ_0 is more had hoc, with no interpretation until now, but we will be able to get rid of this term in the varying θ context. For the time being, we just evaluate these terms to see if they can give account for the supposed dark energy of the standard cosmological model, and if they resolve the cosmological constant problem, that this if they yield a strictly positive value of $-\Lambda$ which is tiny enough to be of the order of the former value preconized by Bennett and al. We first analyze the case of positive constant θ .

13.2 Positive constant Gauss-Bonnet term

From (3.29) and (13.3) we find :

$$3H^2\Omega_{TOT} = R_0^0 - \frac{1}{2}R = \frac{b-3c}{2} \quad (13.7)$$

Equation (8.10) gives the value :

$$\frac{b-3c}{2} = 3 \left(\frac{1}{a^2} + \chi^2 \right) = 3 \left(\frac{1}{a^2} + H^2 \right) \quad (13.8)$$

So we find :

$$\Omega_{TOT} = \frac{1}{a^2} + 1 \quad (13.9)$$

which is an especially interesting solution to the flatness problem. We find that when $t \rightarrow +\infty$, we have $\Omega_{TOT} \rightarrow 1$, which is what is observed, plus the fact that $\Omega_{TOT} > 1$, which gives account for the fact that the observed value is around 1.02 rather than around 1. The term coming from the Gauss-Bonnet expression is, from (5.1) :

$$3H^2\Omega_{GB} = -\theta \frac{b}{3}(b-3c) \quad (13.10)$$

but from (8.1) we obtain

$$\theta \frac{b}{3} = -\frac{1}{4} \quad (13.11)$$

Finally we find :

$$\Omega_{GB} = \frac{1}{2} \Omega_{TOT} \quad (13.12)$$

So, the value we find for the cosmological constant term is just of the right order of magnitude, is explained by the sole quantum corrections to classical general relativity, and is even very near the observed value, taking into account that this value is model dependent. With varying θ and other behaviors of $\kappa(\epsilon)$ than the simple $\kappa(\epsilon) \sim 1/\sqrt{\epsilon}$, we possess a entire new class of cosmological models displaying about the same quantum features. Clearly the model of constant θ , $\theta > 0$, gives already the whole set of features needed for our quantum gravity. It gives a solution to the flatness problem, to the cosmological constant problem, it smoothes out the initial singularity, and contains in one sole quantum equation of gravity the information of the largest and smallest scales of the universe, that is to say the behavior of the radius a of the universe and the state of matter, relativistic or non relativistic, via the pressure p .

13.3 Additional remarks

We analyze now the Λ_0 term : We know, from (11.6), that

$$\Lambda_0 = -\frac{3}{2} H^2 \quad (13.13)$$

So we find :

$$\Omega_{\Lambda_0} = \frac{1}{2} \quad (13.14)$$

This term has again the right sign and is very near the observed value, in the context of the standard cosmological model. If we now look at the values of the three different parts of Ω , we can define $\Omega_\Lambda = \Omega_{GB} + \Omega_{\Lambda_0}$ and compute :

$$\Omega_\Lambda = \frac{1/2 + \dot{a}^2}{1 + \dot{a}^2} \Omega_{TOT} \quad (13.15)$$

We know that in this model, the value of \dot{a}^2 tends to infinity when $t \rightarrow +\infty$. This model predicts a quite interesting situation. In the early universe, $\dot{a}^2 \approx 0$ and

$$\Omega_\Lambda = \frac{1}{2}\Omega_{TOT}$$

Then \dot{a}^2 regularly increases, and the ratio of Ω_Λ to Ω_{TOT} increases also with \dot{a}^2 . We could imagine an actual value of \dot{a}^2 to be around unity : $\dot{a}^2 \approx 1$ such that

$$\Omega_\Lambda = \frac{3}{4}\Omega_{TOT}$$

This does not fit completely because for $\dot{a}^2 \approx 1$, Ω_{TOT} becomes twice too big. This error comes probably from the value of Λ_0 that should disappear in the case of varying θ . Later, the value of \dot{a}^2 should continue to increase and the former ratio will tend to 1 when $t \rightarrow +\infty$:

$$\Omega_\Lambda = \Omega_{TOT}$$

Thus, the universe is evolving to a situation where an apparent dark energy is growing until being almost all of matter. We find a situation analogous to an empty closed and expanding universe.

13.4 The case of constant and negative Gauss-Bonnet term

We first notice that in the proof of the relation

$$\Omega_{GB} = \frac{1}{2}\Omega_{TOT} \tag{13.16}$$

we did not use the specific equations concerning the different signs of θ . Thus this relation is still valid in the negative case. For the term Λ_0 , it is positive from (11.15) or (11.16), and for this reason has the wrong sign. It readily appears that the negative case, a priori, does not fit as easily as the positive case to cosmological observations. Nevertheless we have :

$$\Omega_{TOT} = \frac{1}{3H^2}G_0^0 = \left(\frac{K}{a^2} + H^2\right) = \frac{K}{\dot{a}^2} + 1 \tag{13.17}$$

We find again in this case that Ω_{TOT} should take values near unity, a little greater than 1 in the closed model and a little less than 1 in the open model.

Part IV

The generalized quantum equation of gravity

14 Equations in the general case

14.1 Introduction : the problem of the constant term

As mentioned earlier, we do not have any interpretation for Λ_0 . Until now, this constant has been very useful because the case of constant θ has shown the main features of the quantum equation of gravity, with only very simple calculations. Now, in order to get rid of the problem of the interpretation of Λ_0 , we put it equal to zero. For the computations below, we first leave $\theta(a)$ undetermined, in order to study the different solutions of our equation.

14.2 Computation of the equations

We know that to determine ϵ , there are two equations, equivalent to our quantum problem, up to the constant of integration Λ_0 , which is now zero. These two equations are (5.2) and (5.3) : Equation (5.3) is an equation in which Λ has been eliminated, and is still valid with varying θ , because θ appears only in Λ :

$$\dot{b} - 3\dot{c} = 2\kappa(\epsilon)\dot{\epsilon} = 2\kappa_0 \frac{\dot{\epsilon}}{\sqrt{\epsilon}} \quad (14.1)$$

(5.2) yields :

$$\dot{\theta} \frac{b}{3}(b-3c) + \theta \frac{\dot{b}}{3}(b-3c) + \theta \frac{b}{3}(\dot{b}-3\dot{c}) = \kappa'(\epsilon)\epsilon\dot{\epsilon} = -\kappa_0 \frac{\dot{\epsilon}}{2\sqrt{\epsilon}} \quad (14.2)$$

In order to impose $\Lambda_0 = 0$, we have to use (5.4), as explained in section 5 :

$$\frac{1}{2}(b - 3c) + \Lambda = \kappa(\epsilon)\epsilon = \kappa_0\sqrt{\epsilon} \quad (14.3)$$

We can now impose $\Lambda_0 = 0$ and the definition of Λ becomes, from (5.1) :

$$\Lambda = \theta\frac{b}{3}(b - 3c) \quad (14.4)$$

Using (4.7), (3.32) and (3.33) are equivalent to :

$$b = -\frac{3\ddot{a}}{a} \quad (14.5)$$

and

$$b - 3c = 6\left(\frac{K}{a^2} + H^2\right) \quad (14.6)$$

We know that

$$H = \frac{\dot{a}}{a} \quad (14.7)$$

is no more constant. From (14.3), we can deduce :

$$\frac{1}{2}(b - 3c) - \frac{\ddot{a}}{a}\theta(b - 3c) = \kappa(\epsilon)\epsilon = \kappa_0\sqrt{\epsilon} \quad (14.8)$$

Replacing now the value of $(b - 3c)$, coming from (14.6), in the last relation, we obtain :

$$6\left(\frac{K}{a^2} + H^2\right)\left(\frac{1}{2} - \theta\frac{\ddot{a}}{a}\right) = \kappa(\epsilon)\epsilon = \kappa_0\sqrt{\epsilon} \quad (14.9)$$

We can also integrate equation (14.1) to find, making use of the value of $b - 3c$ again :

$$6\left(\frac{K}{a^2} + H^2\right) = 2\int\kappa(\epsilon)d\epsilon = 4\kappa_0\sqrt{\epsilon} + \lambda_0 \quad (14.10)$$

where λ_0 is a constant of integration. We emphasize that in the last two equations, the first equalities are the exact relations for any $\kappa(\epsilon)$, the second being of course the case

$$\kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\epsilon}}$$

14.3 Checking the dependence of the three equations

We verify, in the case of the last value of $\kappa(\epsilon)$, and as a check, that the three equations (14.2), (14.9) and (14.10) are in fact dependent. We combine the first and third equations (14.2) and (14.10), taking into account that (14.10) and (14.1) are equivalent, and we find :

$$\dot{\theta}\frac{b}{3}(b-3c) + \theta\frac{\dot{b}}{3}(b-3c) + \theta\frac{b}{3}(\widehat{b-3c}) = -\kappa_0\frac{\dot{\epsilon}}{2\sqrt{\epsilon}} = -\frac{1}{4} \times 2\kappa_0\frac{\dot{\epsilon}}{\sqrt{\epsilon}} = -\frac{(\widehat{b-3c})}{4} \quad (14.11)$$

which, for $\theta \neq 0$, gives :

$$\dot{\theta}\frac{b}{3}(b-3c) + \theta\frac{\dot{b}}{3}(b-3c) + \theta\left(\frac{b}{3} + \frac{1}{4\theta}\right)(\widehat{b-3c}) = 0$$

or :

$$\left[\theta\left(\frac{b}{3} + \frac{1}{4\theta}\right)\right](b-3c) + \theta\left(\frac{b}{3} + \frac{1}{4\theta}\right)(\widehat{b-3c}) = 0$$

and we find :

$$\left(\frac{1}{4} + \theta\frac{b}{3}\right)(b-3c) = K_0 \quad (14.12)$$

where K_0 is another constant of integration. We finally find the equation :

$$6\left(\frac{K}{a^2} + H^2\right)\left(\frac{1}{4} - \theta\frac{\ddot{a}}{a}\right) = K_0 \quad (14.13)$$

We are left with three dependent equations (14.9), (14.10) and (14.13), if and only if the following condition on the constants of integration is satisfied :

$$K_0 = -\frac{1}{4}\lambda_0 \quad (14.14)$$

To see this, we take (14.9) and add to it equation (14.10) multiplied by $-1/4$. This result corresponds to what we wanted to prove : the three equations are dependent, but the constants of integration can no longer be taken at will.

15 The three equations of quantum cosmology

15.1 The three equations

We now summarize the former calculations to write down the system of the three equations of movement, which are equivalent to the whole set of equations describing quantum gravity. In the former equation, we take the constants K_0 and λ_0 equal to zero to obtain only the simplest solution:

$$\theta(a)\frac{\ddot{a}}{a} = \frac{1}{4} \quad (15.1)$$

and also :

$$\frac{K}{a^2} + H^2 = \frac{2}{3}\kappa_0\sqrt{\epsilon} \quad (15.2)$$

We do not forget equation (5.5) which gives us the pressure p :

$$\frac{b+c}{2} - \Lambda = \kappa(\epsilon)p \quad (15.3)$$

15.2 The equation for the pressure

We start from the former relation, taking into account that

$$\kappa(\epsilon) = \frac{\kappa_0}{\sqrt{\epsilon}}$$

and also from (14.1), (14.5) and (15.1) :

$$\Lambda = \theta\frac{b}{3}(b-3c) = -\frac{\theta\ddot{a}}{a}(b-3c) = -\frac{1}{4}(b-3c)$$

Using equation (15.3) for p , we obtain :

$$\frac{b+c}{2} - \Lambda = \frac{b+c}{2} + \frac{b-3c}{4} = \frac{3b-c}{4} = \frac{\kappa_0}{\sqrt{\epsilon}}p \quad (15.4)$$

We now use the identity :

$$\frac{3b-c}{4} = \frac{1}{12}(b-3c) + \frac{2}{3}b$$

Using equations (14.5) and (14.6) for the values of b and $b-3c$:

$$b = \frac{-3\ddot{a}}{a}$$

and

$$b - 3c = 6 \left(\frac{K}{a^2} + H^2 \right)$$

we find :

$$\frac{1}{2} \left(\frac{K}{a^2} + H^2 \right) - \frac{2\ddot{a}}{a} = \frac{\kappa_0}{\sqrt{\epsilon}} p \quad (15.5)$$

We now use the other two equations. From (15.2) we have :

$$\frac{1}{2} \left(\frac{K}{a^2} + H^2 \right) = \frac{1}{3} \kappa_0 \sqrt{\epsilon} \quad (15.6)$$

From (15.1) we have :

$$-\frac{2\ddot{a}}{a} = -\frac{1}{2\theta} \quad (15.7)$$

We combine all these relations to obtain :

$$\frac{1}{3} \kappa_0 \sqrt{\epsilon} - \frac{1}{2\theta} = \frac{\kappa_0}{\sqrt{\epsilon}} p \quad (15.8)$$

and finally :

$$p = \frac{\epsilon}{3} - \frac{\kappa_0^{-1} \sqrt{\epsilon}}{2\theta} \quad (15.9)$$

We deduce from this equation that :

$$p = \frac{\epsilon}{3} \left(1 - \frac{1}{\frac{2}{3} \kappa_0 \sqrt{\epsilon} \theta(a)} \right) \quad (15.10)$$

and also, using (15.2) :

$$p = \frac{\epsilon}{3} \left(1 - \frac{1}{\left(\frac{K}{a^2} + H^2 \right) \theta(a)} \right) \quad (15.11)$$

We prove in the next section that there exists a function $\theta(a)$ which gives the relation $p = \epsilon/3$ in the early universe and gives the other relation $p = 0$ for the present universe, as it should.

15.3 Behavior of the pressure

We notice here that different choices of θ in the quantum equation give different values for the pressure p , and especially we can obtain any behavior of the kind

$$p = \lambda \epsilon \quad (15.12)$$

for any value of λ , such that $0 \leq \lambda \leq \frac{1}{3}$, provided we make the right choice of θ . Indeed, from (15.9) and (15.12), we find :

$$\left(\frac{1}{3} - \lambda\right) \epsilon = \left(\frac{1 - 3\lambda}{3}\right) \epsilon = \frac{\kappa_0^{-1} \sqrt{\epsilon}}{2\theta}$$

or :

$$\frac{1}{\theta} = (1 - 3\lambda) \left(\frac{2}{3} \kappa_0 \sqrt{\epsilon}\right) = (1 - 3\lambda) \left(\frac{K}{a^2} + H^2\right) \quad (15.13)$$

For example the case $p = 0$ is obtained for the choice :

$$\theta = \frac{a^2}{K + \dot{a}^2} \quad (15.14)$$

It appears that there are three natural choices for the behavior of θ : proportional to a^2 , inversely proportional to H^2 or inversely proportional to $\kappa_0 \sqrt{\epsilon}$. These three choices display different behavior of the pressure p . We have just seen that the last choice displays a behavior of the kind

$$p = \lambda \epsilon$$

with constant λ . The choice

$$\frac{\theta_1}{H^2} \quad (15.15)$$

leads to

$$p = \frac{\epsilon}{3} \left(1 - \frac{\dot{a}^2}{\theta_1(1 + \dot{a}^2)}\right) \quad (15.16)$$

If we take the value $\theta_1 = 1$, the equation

$$\frac{\ddot{a}}{a} = \frac{1}{4\theta(a)} = \frac{\dot{a}^2}{4a^2} \quad (15.17)$$

has a solution $a(t) = kt^\alpha$, because putting this value of $a(t)$ in the equation, we obtain identically : $\alpha(\alpha - 1) = \alpha^2/4$ and $\alpha = 4/3$. We thus find : $\dot{a} \rightarrow 0$ for the early universe and $\dot{a} \rightarrow +\infty$ for the late universe. In these conditions, formula (15.16) gives $p = \epsilon/3$ for the early universe and $p = 0$ for the late universe. In fact, the behavior of p in this case is not completely satisfying, because looking more precisely at the formula for p , we see that even if p tends to zero when $t \rightarrow +\infty$, it is not decreasing to zero fast enough. For example, the value $\dot{a} = 1$ gives only $p = \epsilon/6$. Still, we notice here that concerning these problems, we can improve the predictions of our model by changing the behavior of two functions : $\theta(a)$ and $\kappa(\epsilon)$.

Part V

Generalized equation : the quantum solution to the cosmological problems

In this part, we recall that we still take a varying θ , and at the same time, we impose the condition $\Lambda_0 = 0$. So the Λ term has now a completely determined origin : it represents the exact Gauss-Bonnet term. In this context, we show how the quantum equation of gravity can solve a number of cosmological problems, by a judicious choice of θ . We first prove that the sign of θ should be positive, that the expansion problem has always a solution, for any choice of θ . We then prove that the flatness and cosmological constant problems find also their solutions from the quantum equation. We finally recall that our equation of quantum gravity leads to three equations for cosmology, (15.1), (15.2) and (15.9). (15.1) gives the behavior of the parameter a :

$$\theta(a) \frac{\ddot{a}}{a} = \frac{1}{4} \quad (15.18)$$

The second equation (15.2) gives the relation between Hubble's constant, the radius a of the universe, and the energy density ϵ :

$$\frac{K}{a^2} + H^2 = \frac{2}{3} \kappa_0 \sqrt{\epsilon} \quad (15.19)$$

The last equation (15.9) gives the pressure p :

$$p = \frac{\epsilon}{3} - \frac{\sqrt{\epsilon}}{2\kappa_0\theta} \quad (15.20)$$

16 Constraints on the Gauss-Bonnet parameter and the expansion problem

16.1 Returning to the formula for the pressure

From the former relation

$$p = \frac{\epsilon}{3} - \frac{\sqrt{\epsilon}}{2\kappa_0\theta}$$

giving the value of the pressure p , we can give an important constraint on $\theta(a)$. As already noticed, the former equation for p proves clearly that the value of p depends essentially on the value of $\theta(a)$, so an appropriate choice of θ can give us the value we need or want for p . Furthermore we see that the value

$$\frac{1}{\theta} = \frac{2}{3}\sqrt{\epsilon} = \frac{K}{a^2} + H^2 \quad (16.1)$$

gives $p = 0$ exactly. We also see that the quantum theory possesses a very interesting limit, which is $\theta \rightarrow +\infty$. In this case the particles are made of perfect relativistic stuff, since then we have the exact relation :

$$p = \frac{\epsilon}{3} \quad (16.2)$$

16.2 The constraint on the Gauss-Bonnet parameter

We reanalyzed the formula for p because we also have on p the constraint :

$$0 \leq p \leq \frac{\epsilon}{3}$$

The second inequality, compared to the value of p , leads to

$$\theta(a) > 0 \quad (16.3)$$

This sign of θ is a general condition which should always be valid in the context of the quantum equation of gravity. We notice that if in the context of constant θ , we have been able to study the case $\theta < 0$, this was only because we supposed a non vanishing Λ_0 . We are now in the case $\Lambda_0 = 0$, and for this reason a negative sign for θ is no longer possible.

16.3 The expansion problem

We recall that the expansion problem shall be solved once explained why our universe is in expansion. In other words, solving the problem is explaining, independently of any choice of initial conditions for \dot{a} in the early days of the universe, why we have presently the condition $\dot{a} > 0$. We now turn to the relation :

$$\theta \frac{\ddot{a}}{a} = \frac{1}{4}$$

and see that, since $\theta > 0$, and of course, since $a > 0$, we have $\ddot{a} > 0$, and \dot{a} is an increasing function of time. This explains its positivity with no reference to initial conditions, as proved below.

16.4 Asymptotic behavior of the time derivative of the radius

An increasing function like $\dot{a}(t)$ is not, mathematically, necessarily positive. But, if we suppose that the universe is necessarily old, the values of t in our equations must be large. We can suppose, mathematically, for this reason, that the present regime corresponds to $t \rightarrow +\infty$. We just concluded that $\theta(a) > 0$ and $\ddot{a}(t) > 0$. Thus, $\dot{a}(t)$ is strictly increasing. An increasing function will necessarily have a limit λ , when $t \rightarrow +\infty$. If $\lambda \neq \pm\infty$, we know that we have, when $t \rightarrow +\infty$:

$$a(t) \sim \lambda t \tag{16.4}$$

where the symbol $f(t) \sim g(t)$ is used, here, to signify that the ratio of these two functions tends to 1 when $t \rightarrow +\infty$. When $\lambda < 0$, then, in the $t \rightarrow +\infty$ regime, $a(t) < 0$ which is physically impossible. When $\lambda = -\infty$ the situation is even worse. So we conclude that $\lambda \geq 0$. If we discard the case $\lambda = 0$, we are left with $\lambda > 0$, and to reach this strictly positive limit, the function $\dot{a}(t)$ has to be strictly positive in the $t \rightarrow +\infty$ regime. We could even discard the case $\lambda = 0$ by mathematical arguments. Indeed, we prove just below that in fact the value of λ , more than a kind of initial condition for $t \rightarrow +\infty$, is essentially determined by θ itself. This is another feature of the quantum equation that initial conditions are completely determined by the equation itself, and this is another

sign of its quantum nature. Indeed, we know that the Heisenberg's uncertainty relations are based on the principle that to go from the classical theory to the quantum theory, the classical concept of initial conditions has to be abandoned, as analyzed Landau.

16.5 The role of the Gauss-Bonnet parameter

Let us suppose here, to fix ideas, that $\lambda > 0$, or $\lambda = +\infty$. We have

$$\lim_{t \rightarrow +\infty} a(t) = \lambda$$

and we pose

$$\mu = \frac{\lambda^2}{\lambda^2 + K}$$

Recalling that $K = +1$ and $K = -1$ correspond respectively to the closed and open models, we have $0 < \mu \leq 1$ in the closed model and $\mu \geq 1$, or $\mu < 0$, in the open model. We can adopt the convention that $\mu = 1$ in the case $\lambda = +\infty$, and this in both models.

We use again the equation :

$$\theta \frac{\ddot{a}}{a} = \frac{1}{4}$$

or equivalently :

$$\ddot{a} = \frac{a}{4\theta}$$

Multiplying this relation by \dot{a} we find :

$$\ddot{a}\dot{a} = \frac{a\dot{a}}{4\theta} \tag{16.5}$$

and after integration, with \dot{a}_0 taken as a constant of integration :

$$\dot{a}^2 = \dot{a}_0^2 + \int_{a_0}^a \frac{ada}{2\theta(a)} \tag{16.6}$$

where the condition for λ finite is :

$$\int_{a_0}^{+\infty} \frac{ada}{2\theta(a)} < +\infty \tag{16.7}$$

reminding ourselves that $\theta(a) > 0$. We now make the hypothesis that, when $t \rightarrow -\infty$, the universe is inflationary, so at this value of t , it can be written that $a_0 = \dot{a}_0 = 0$, and we obtain :

$$\lambda = \int_0^{+\infty} \frac{ada}{2\theta(a)} \tag{16.8}$$

This proves that with the additional assumption of inflation in the early universe, which was our first postulate, λ is only a characteristic of $\theta(a)$, and is completely determined by our equation. For the time being, it can be simply noticed that postulating inflation is also a kind of initial condition, and that λ has in fact a double nature. A part of it, too, having something to do with a constant of integration. We also precise that when

$$\int_{a_0}^{+\infty} \frac{ada}{2\theta(a)} = +\infty$$

this time $\lambda = +\infty$ and this relation is independent of any initial condition : in this case λ is entirely determined by $\theta(a)$. In this case, the condition $\lambda = 0$ has been completely discarded mathematically.

17 The age of the universe

17.1 Introduction

We want to study the condition $tH \sim 1$ as $t \rightarrow +\infty$. We know that the relation $t \approx 1/H$ is observed, in the context of the standard model of cosmology (Bennett and al., 2003, [2]). The situation for our quantum equation is quite different. In the standard model, there are only a few fixed parameters, and observations of the cosmic background radiation have strong implications on the other predictions of the model. In the quantum regime, we have an entire function $\theta(a)$ to be determined, which leaves much more possibilities. This is because the difficulty has been displaced. With the choice of a entire function, it is easy to explain a lot of phenomena, but the difficulty, in the quantum regime, is that the function $\theta(a)$ also should explain how unification comes into play in the picture. How this is done, is the subject of [57]. In any case, it is interesting to see, at least as an exercise, if there are choices of θ that yield the relation $tH \sim 1$ when $t \rightarrow +\infty$.

17.2 The finite case

We suppose that $0 < \lambda < +\infty$ and we take $t \rightarrow +\infty$, so we know that $\dot{a} \sim \lambda$ and that

$$H = \frac{\dot{a}}{a} \sim \lambda a^{-1}$$

Now integrating $\dot{a} \sim \lambda$ in respect to t , we find $a(t) \sim \lambda t$ so we find $tH \sim 1$ as wanted. Thus, any finite value of λ yields directly the observed relation.

17.3 The infinite case

We have the relation :

$$\dot{a}^2 = \dot{a}_0^2 + \int_{a_0}^a \frac{x dx}{2\theta(x)} \quad (17.1)$$

We study the case : $\theta(a) \sim K_{n,p} a^n (\ln a)^p$ when $a \rightarrow +\infty$.

17.3.1 The case : n strictly greater than 2

In order to have an infinite λ , the integral must be divergent when $a = +\infty$, so finite λ implies the relation $n \leq 2$, and $p \leq 1$ if $n = 2$. We recall that in any other case, that is to say for $n > 2$ and for $n = 2$ with $p > 1$, λ is finite and the condition $tH \sim 1$ is verified from the section 17.2.

17.4 The case : n strictly less than 2

To simplify the calculation, we stick to the case $p = 0$. So it remains the relation $\theta(x) \sim K_n x^n$ when $x \rightarrow +\infty$, and $\lim_{r \rightarrow +\infty} \dot{a}(t) = +\infty$, because λ is infinite. It yields

$$\dot{a}^2 = \dot{a}_0^2 + \int_{a_0}^a \frac{x dx}{2\theta(x)} \sim \int_{a_0}^a \frac{x dx}{2K_n x^n}$$

Calculating the integral, we find :

$$\dot{a}^2 \sim \frac{a^{2-n}}{2(2-n)K_n}$$

We thus have :

$$a^{\frac{n-2}{2}} \dot{a} \sim \frac{1}{\sqrt{2(2-n)K_n}} \quad (17.2)$$

and :

$$a^{n/2} H \sim \frac{1}{\sqrt{2(2-n)K_n}} \quad (17.3)$$

In the case $n < 0$, we obtain, as $a \rightarrow +\infty$, the condition $H \rightarrow +\infty$, ruled out by the small observed value of H . The case $n = 0$ is more interesting, it is analogous to constant positive θ , which we have already studied. Sticking to the case $n > 0$, and integrating our equation we find :

$$\frac{2}{n} a^{\frac{n}{2}} \sim \frac{t}{\sqrt{2(2-n)K_n}}$$

Finally

$$tH \sim \frac{2}{n} \sqrt{2(2-n)K_n} a^{n/2} H \sim \frac{2}{n} \quad (17.4)$$

which rules out combinations $n < 2; p = 0$, for the condition $tH \sim 1$ is to be verified. Of course, for values of n just a little less than 2, we have an approximate relation, since in these cases $2/n \approx 1$. We notice that the former calculations seem to designate a special value of n of particular interest : to obtain $tH \sim 1$, the former calculation under the hypothesis $n < 2$ led us back to the value $n = 2$, which for this reason appears as a kind of central candidate. This value $n = 2$ is also the value we can guess from dimensional arguments, recalling θ has the dimension of a squared length.

17.5 The case n=2

We now suppose that $\theta(x) \sim Kx^2$ when $x \rightarrow +\infty$, and find that :

$$\dot{a}^2 = \dot{a}_0^2 + \int_{a_0}^a \frac{x dx}{2\theta(x)} \sim \int_{a_0}^a \frac{dx}{2Kx}$$

calculating the integral we find :

$$\dot{a}^2 \sim \frac{1}{2K} \ln a$$

This yields

$$\dot{a} \sim \frac{1}{\sqrt{2K}} \sqrt{\ln a} \quad (17.5)$$

or :

$$\frac{\dot{a}}{\sqrt{\ln a}} \sim \frac{1}{\sqrt{2K}}$$

and integrating this equation :

$$\int_{a_0}^a \frac{dx}{\sqrt{\ln x}} \sim \frac{t}{\sqrt{2K}}$$

Calculating this integral by the change of variables : $y = \sqrt{\ln x}$, we find :

$$\int_{a_0}^a \frac{dx}{\sqrt{\ln x}} = 2 \int_{\sqrt{\ln a_0}}^{\sqrt{\ln a}} e^{y^2} dy$$

and we use the relation :

$$\int_{\sqrt{\ln a_0}}^X e^{y^2} dy \sim \int_1^X e^{y^2} dy \sim \frac{e^{X^2}}{2X}$$

To prove the last relation, we notice that it can be written

$$\int_1^X e^{y^2} dy = \int_1^X \frac{2ye^{y^2}}{2y} dy$$

and integrating by parts, integrating $2ye^{y^2}$ and deriving $(2y)^{-1}$, we find :

$$\int_1^X e^{y^2} dy = \frac{e^{X^2}}{2X} - \frac{e}{2} + \int_1^X \frac{e^{y^2}}{2y^2} dy$$

the second integral on the right being negligible compared to the first on the left. The term $e/2$ is negligible too, because it is finite compared to integrals which tend to infinity.

So we obtain the relation :

$$\frac{t}{\sqrt{2K}} \sim \int_{a_0}^a \frac{dx}{\sqrt{\ln x}} = 2 \int_{\sqrt{\ln a_0}}^{\sqrt{\ln a}} e^{y^2} dy \sim \frac{a}{\sqrt{\ln a}} \quad (17.6)$$

We finally obtain, using (17.5) and (17.6) :

$$tH = t \frac{\dot{a}}{a} \sim \sqrt{2K} \frac{a}{\sqrt{\ln a}} \times \frac{1}{a} \times \frac{1}{\sqrt{2K}} \sqrt{\ln a} = 1 \quad (17.7)$$

which proves that in the case $n = 2; p = 0$, the observed relation between H and t is satisfied. The former calculation is another hint that the case $n = 2$ should be preferred. However, this central case is the sole case which exhibits the condition $tH \sim 1$, and $\lambda = +\infty$ together. At the same time, it does not imply that the present value of $\dot{a}(t)$ is

much greater than a number of the order of unity. Despite the fact that $\dot{a} \rightarrow +\infty$ when $t \rightarrow +\infty$, the relation

$$\dot{a} \sim \frac{1}{\sqrt{2K}} \sqrt{\ln a}$$

shows that \dot{a} tends to its limit slowly enough to be even today very far from having taken great values. This case also gives us hints on how could work the equation in the very early universe, a fact that is explained just below.

18 The big bang and before

18.1 The direct calculation

Formerly, we integrated the relation

$$\theta \frac{\ddot{a}}{a} = \frac{1}{4}$$

to obtain

$$\dot{a}^2(t) = \dot{a}_{pr}^2 + \int_{a_{pr}}^{a(t)} \frac{x dx}{2\theta(x)} = \dot{a}_0^2 - \int_{a(t)}^{a_{pr}} \frac{x dx}{2\theta(x)} \quad (18.1)$$

Here, we can take for a_{pr} the present value of the radius of the universe, and t_{pr} is the present value of the cosmological time. Looking back in time, when the value of $a(t)$ was much smaller, we see that the former relation imposes at any time :

$$\int_{a(t)}^{a_{pr}} \frac{x dx}{2\theta(x)} \leq \dot{a}_{pr}^2 \quad (18.2)$$

because the square $\dot{a}^2(t) \geq 0$. Now, for values of $\theta(x)$ such that, for fixed α , the integral

$$\int_0^\alpha \frac{x dx}{2\theta(x)} \quad (18.3)$$

diverges in the vicinity of $x = 0$, the integral in (18.3) tends to infinity, and (18.2) can no longer be verified. So, for these choices of $\theta(x)$, the initial singularity is smoothed out even more drastically than by an exponential growth of $a(t)$. The universe seems to have started with a strictly positive radius, that we note a_0 from now on. In fact a problem immediately arises. Going back in time, has this smallest value of the radius of

the universe been reached in finite or infinite time? If it has been reached in finite time, the value a_0 is only a minimum of the function $a(t)$, and we have to suppose that the universe started with an infinite value of $a(t)$ for $t \rightarrow -\infty$, then reduced to the minimum value a_0 , and then grew again to give the universe we know. All these conclusions are based on the relation $\ddot{a} > 0$. If the smallest value a_0 is reached in infinite time, we have no beginning for the universe either, but now with the picture of a since ever growing universe, from a radius a_0 at the time $t \rightarrow -\infty$. We take the value $\theta(x) = \theta_0 x^2$, where θ_0 is a constant. We see that

$$\int_0^\alpha \frac{x dx}{2\theta(x)}$$

diverges in the vicinity of $x = 0$, so we find that there is a smallest possible radius a_0 . In both cases, if this value is only a limit when $t \rightarrow -\infty$, or if this value is only a minimum of $a(t)$ for a finite value t_0 , we find that $\dot{a}_0 = 0$. So the relation for $a(t)$ reads :

$$\dot{a}^2(t) = \int_{a_0}^{a(t)} \frac{x dx}{2\theta(x)} \quad (18.4)$$

It is then a simple exercise to find that the value a_0 have been reached in finite time. In fact there is no choice of $\theta(x)$ that can change this general fact. Indeed, we know that

$$\ddot{a} = \frac{a}{4\theta(a)}$$

When approaching the value a_0 , we can suppose that $\theta(x)$ is an increasing function of x , even in the general case where it is only supposed that the integral (18.3) diverges. Indeed, for the integral to diverge in the vicinity of $x = 0$, the ratio $\theta(x)/x$ has to tend to zero, so it is natural to suppose that $\theta(x)$ is an increasing function of x , at least for small values of x , and $x = a_0$ is supposed to be a small value of x . When we go to the value a_0 , the function θ goes to its minimum value $\tilde{\theta}_0$. For example $\tilde{\theta}_0 = \theta_0 a_0^2$ in the particular $\theta(x) = \theta_0 x^2$ case. We thus see that \ddot{a} tends to the finite strictly positive value $a_0/4\tilde{\theta}_0$. If the value a_0 were reached for $t \rightarrow -\infty$, we would have seen \ddot{a} tend to zero, which is impossible. So for any function $\theta(x)$, a_0 is only a minimum reached in finite time. The conclusion is that our universe had a shrinking phase from $a = +\infty$ when $t = -\infty$ to our big bang, $a = a_0$ and $t = 0$, and is in a expanding phase since then and for ever. We will study more precisely what could have happened before the big bang in [57], but we

should be aware that no definitive conclusion can be made about this period. The general principle of logic is that it is never possible to conclude in a region of knowledge where we cannot be contradicted by experiment. As an exercise, we imagine in the next section a physical principle, which cannot either be contradicted by experiment, and which yields opposite conclusions about the big bang.

18.2 An indirect calculation

We suppose now that the function $\theta(x)$ possesses discrete values. In other words, we make an additional hypothesis on θ , analogous to the statement that permits to go from classical values of the energy to quantum values : in classical physics, the energy takes continuous positive values, whereas in quantum physics, there is a mass gap, the first values of the energy being quantized. So we suppose that $\theta(a)$ is defined by a kind of approximate formula which still is : $\theta(a) = \theta_0 a^2$, analogous to the classical continuous values of the energy, but that this formula has to be furthermore corrected, by quantizing the values of θ , $\tilde{\theta}_0$ being its smallest strictly positive value, and we note $\tilde{\theta}_1$ the smallest value of $\theta(x)$ strictly greater than $\tilde{\theta}_0$. What does now happen for the parameter a in the very early universe? The approximate value of θ shows as we said that going back in time the radius is shrinking to the value a_0 , which is a minimum. Now we take into account the true discrete values of θ , which, as we can see, makes θ become a step function. We see that as the radius is shrinking, the discrete values of θ are going smaller. When $a(t)$, which tends to a_0 , becomes strictly less than a_1 corresponding to $\tilde{\theta}_1$, which means $\tilde{\theta}_1 = \theta_0 a_1^2$, the value of θ becomes definitively equal to $\tilde{\theta}_0$, and our universe becomes definitively exponentially growing, with a constant value of Hubble's constant. This results from our calculations of the constant and strictly positive θ case of Part III. So we find another principle, which changes enough the behavior of θ , to make the big bang look completely different, in such a manner that no sure conclusion can be made on this matter which stays unreachable by experiments.

19 The cosmological constant problem

19.1 The problem in the classical context

As far as the standard model of cosmology is concerned, the cosmological parameters of the model are measured with a very good approximation (Bennett and al. 2003, [2]). In particular there are, in this model, two important parameters, the total energy density $\Omega \approx 1$ and the energy density of dark matter Ω_Λ , the observed relation being :

$$\Omega_\Lambda = \frac{3}{4}\Omega \quad (19.1)$$

There is a lot of dark energy density, which remains unexplained. Furthermore, the model uses the equation of general relativity, with a cosmological constant Λ :

$$R_{ik} - \frac{1}{2}Rg_{ik} - \Lambda g_{ik} = 8\pi GT_{ik} \quad (19.2)$$

Now the term Ω_Λ is defined by the formula :

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \approx \frac{3}{4} \quad (19.3)$$

The cosmological constant problem is to understand why a constant like Λ should be nonzero, and furthermore should possess such a tiny strictly positive value :

$$\Lambda \approx \frac{9H^2}{4} \quad (19.4)$$

Finally, we can state the problem in the following way : the equations of general relativity are in the number of two, one which gives ϵ , the other gives p . We have, furthermore, the equation of conservation of entropy, so three equations plus the fact that they are dependent. So we choose two equations, say the conservation of entropy and :

$$R_0^0 - \frac{1}{2}R - \Lambda = 8\pi G\epsilon \quad (19.5)$$

If we pass the constant Λ to the right hand side of the equation, and insert it in the term in ϵ , we find a new ϵ , which we could call ϵ_{app} , because it is an apparent energy density. The value of Λ is such that :

$$\epsilon_{app} = 4\epsilon \quad (19.6)$$

As a remark, ϵ_{app} is the value of the observed energy density, when the equation without the cosmological term is used, that is to say we have

$$R_0^0 - \frac{1}{2}R = 8\pi G\epsilon_{app}$$

The other equation just gives the relation between p and ϵ . If we want this conservation of entropy still to be valid for apparent quantities, we have to pose

$$p_{app} = 4p$$

but since in the case of the standard model we have $p = 0$, this does not change anything for the value of the pressure.

19.2 The quantum equation

In the context of the quantum equation, we know the origin of the Λ term. We know from (5.1) that :

$$\Lambda = \theta(a)\frac{b}{3}(b - 3c) + \Lambda_0 \quad (19.7)$$

where we have supposed $\Lambda_0 = 0$, so the whole Λ term has an identified origin : it corresponds to topological corrections to classical gravity. We recall that we had :

$$b = \frac{-3\ddot{a}}{a}$$

and

$$\frac{\ddot{a}}{a} = \frac{1}{4\theta}$$

So we find that

$$\Lambda = -\frac{b - 3c}{4} \quad (19.8)$$

whereas the value of $G_0^0 = R_0^0 - \frac{1}{2}R$ can be read in (3.29) and (14.6) :

$$G_0^0 = \frac{b - 3c}{2} = 3 \left(\frac{K}{a^2} + H^2 \right) \quad (19.9)$$

In the quantum context, we also have two equations, plus the conservation of entropy, and they also are dependent. We can choose equation (19.9) and the conservation of entropy.

Then, we have to analyze (19.9), and how ϵ is affected by forgetting the Λ Gauss-Bonnet term. We compute, using (19.8), (19.9) and (15.2) :

$$G_0^0 + \Lambda = \frac{b - 3c}{4} = \frac{3}{2} \left(\frac{K}{a^2} + H^2 \right) = \kappa_0 \sqrt{\epsilon}$$

If we forget the Λ term in this equation, we have to replace

$$G_0^0 + \Lambda = \kappa_0 \sqrt{\epsilon} \tag{19.10}$$

by :

$$G_0^0 = \kappa_0 \sqrt{\epsilon_{app}} \tag{19.11}$$

where ϵ_{app} is the apparent matter density, exactly as we did in our analysis of the case of general relativity. The difference is that now ϵ_{app} can be calculated from the quantum equations and compared to the original ϵ . Equations (19.8) and (19.9) give directly :

$$\Lambda = -\frac{1}{2}G_0^0 \tag{19.12}$$

Forgetting Λ in our equation would have the net effect of changing

$$G_0^0 + \Lambda = \frac{1}{2}G_0^0$$

for G_0^0 . So we see that the net effect of forgetting the Λ -term on the left hand side of the equation is to multiply the right hand side by 2, which has the effect of doubling κ_0 , if we interpret this change in terms of a change of the gravitational constant. However, if we prefer interpret the change in the equation as a change in ϵ , we get the right relation :

$$\epsilon_{app} = 4\epsilon \tag{19.13}$$

and as already noticed, ϵ_{app} is the new apparent matter density. So the lack of the Λ -term in our equation makes us see a density four times bigger than it should. This factor 4 corresponds to a prediction of the quantum equation, and is equal to the factor 4 coming from the observations of the cosmos, in the context of general relativity. This is striking enough to make us think that we are on the right track with our equation of quantum gravity.

19.3 Complete calculation of the cosmological constant

If we compute Ω_Λ by the method of section 13.1, which uses the fact that our observations of the values of the masses in the cosmos are only based on the principle of inertia, we obtain the relation (13.3) :

$$\Omega_{TOT} = \frac{G_0^0}{3H^3} \quad (19.14)$$

We recall that we had (19.12) :

$$\Lambda = -\frac{1}{2}G_0^0 \quad (19.15)$$

and from (19.9) and (15.2) :

$$G_0^0 = 3 \left(\frac{K}{a^2} + H^2 \right) = 2\kappa_0\sqrt{\epsilon} \quad (19.16)$$

So we obtain :

$$\Omega_{TOT} = \frac{2\kappa_0\sqrt{\epsilon}}{3H^2} \quad (19.17)$$

We know that Λ is negative because it possesses an extra minus sign compared to the usual Λ of general relativity. Putting all these relations together, we find that our equation predicts for the usual Λ a positive value, verifying :

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = \frac{1}{2}\Omega_{TOT} \quad (19.18)$$

which is clearly in the domain of uncertainties of the observations, since this domain is determined by the relations

$$-1 < \frac{\Lambda}{3H^2} < 2$$

With $\Omega = 1.02$, our Λ is just at the center of the former interval.

19.4 A remark on the coefficients 2 and 4 of the former sections

We now observe that the coefficient 4 between the true physical and apparent energy densities is only 4 because it is viewed from the place of ϵ , under the square root. Of course this coefficient becomes 2, viewed from the place of κ_0 , or even from the place of Λ , that is to say outside the square root. The interpretation of this factor 4 depends on

how the quantum equation is established in the context of unification, and depends on the origin of the dependence of the gravitational coupling G on ϵ . Here we just rapidly explain how things could go, in a complete unified theory. When we double κ_0 , reestablishing \hbar and c , we double in fact $\kappa_0/\sqrt{\hbar c}$. Now, suppose that in a unified theory, multiplying the gravitational constant by some factor has the effect of multiplying also \hbar by the same factor. In the former section, we saw that the effect of Λ was to multiply, not κ_0 , but $\kappa_0/\sqrt{\hbar c}$, by 2. Given our hypotheses, to multiply this term by 2, we have to multiply κ_0 by 4, such that, \hbar being multiplied by 4, the complete ratio $\kappa_0/\sqrt{\hbar c}$ is only multiplied by 2. So it can be seen that in the coefficient 4 multiplying the energy density, and coming from the Gauss-Bonnet term, there is most probably a factor 2 which is a classical correction to ϵ , and another factor 2 coming from further, more fundamental, quantum corrections to 2ϵ . Or in other words, there is a factor 2 coming from the quantum corrections due to Λ , and another factor 2 coming from corrections belonging to unification.

20 The flatness problem

20.1 Value of the time derivative of the radius of the universe

We have seen that, because \dot{a} is an increasing function of time, it has to possess a limit λ when $t \rightarrow +\infty$, where $\lambda > 0$ is finite or infinite. Furthermore, we have shown that in the case $\theta(a) = \theta_0 a^2$, the relation is (17.5) :

$$\dot{a} \sim \frac{1}{\sqrt{2\theta_0}} \sqrt{\ln(a/a_0)}$$

such that the present value of \dot{a} is still finite, because even if $\lambda = +\infty$, \dot{a} goes so slowly to infinity, that it should, in the present universe, take its value around unity. In any case, we wrote

$$\mu = \frac{\lambda^2}{\lambda^2 + K} \quad (20.1)$$

with $0 \leq \mu \leq 1$, for the closed model, which is characterized by the relation $K = +1$, and $\mu \geq 1$ in the open model corresponding to $K = -1$. We adopt the convention that $\mu = 1$

in the case $\lambda = +\infty$ for both models. However, from now on, we note λ the present value of \dot{a} , and as we said λ should be around unity.

20.2 The classical and quantum flatness problems

We want to prove that the total energy density of matter, Ω_{TOT} , that is to say the energy density when dark energy is taken into account, at least has a present value near unity. In classical gravity, the value of Ω_{TOT} takes the form

$$\Omega_{TOT} = \frac{8\pi G\epsilon_{app}}{3H^2} \quad (20.2)$$

In the quantum regime, a reasonable relation between G and κ_0 is found by comparing general relativity and quantum gravity, where we now have :

$$\frac{K}{a^2} + H^2 = \frac{2}{3}\kappa_0\sqrt{\epsilon} \quad (20.3)$$

In general relativity the relation was :

$$\frac{K}{a^2} + H^2 = \frac{8\pi G}{3}\epsilon + \frac{\Lambda}{3} \quad (20.4)$$

as can be seen for example in Peebles, 1993, [49], equation (5.18). As we know that the contribution of Λ is about three quarters of the total energy density, sticking on the true ϵ , we find :

$$\frac{K}{a^2} + H^2 = \frac{32\pi G}{3}\epsilon \quad (20.5)$$

Of course, this relation also results from the relation $\epsilon_{app} = 4\epsilon$, which is a consequence of the quantum equation of gravity. We thus should have :

$$\kappa_0 = 16\pi G\sqrt{\epsilon} \quad (20.6)$$

This relation can also be proved by using the principle of equivalence between gravitation and inertia. Using this principle we led us to (19.16), the quantum expression for Ω_{TOT} was found in (19.19). We thus obtain :

$$\Omega = \frac{8\pi G\epsilon_{app}}{3H^2} = \frac{32\pi G\epsilon}{3H^2} = \frac{2\kappa_0\sqrt{\epsilon}}{3H^2} \quad (20.7)$$

and we find (20.6) again. We have to prove that this expression of Ω_{TOT} tends to a finite value when $t \rightarrow +\infty$. We know that the present value of \dot{a} is $\dot{a} = \lambda$. We thus find

$$\frac{1}{a^2} = \frac{H^2}{\lambda^2}$$

Finally, we obtain :

$$\frac{1}{\mu} H^2 = \left(\frac{K}{\lambda^2} + 1 \right) H^2 = \left(\frac{K}{\dot{a}^2} + 1 \right) H^2 = \frac{K}{a^2} + H^2 = \frac{2}{3} \kappa_0 \sqrt{\epsilon}$$

such that :

$$\frac{1}{\mu} H^2 = \frac{2}{3} \kappa_0 \sqrt{\epsilon} \quad (20.8)$$

We now use the value of Ω_{TOT} to find :

$$\Omega = \Omega_{TOT} = \frac{2\kappa_0 \sqrt{\epsilon}}{3H^2} = \frac{1}{\mu} \geq 1 \quad (20.9)$$

Here we suppose that we are in the closed model. Indeed, we have

$$\frac{1}{\mu} = \frac{K}{\lambda^2} + 1 \quad (20.10)$$

in such a way that $1/\mu \approx 1$ and $1/\mu \geq 1$ if and only if we are in the closed model. Since the observed value of Ω_{TOT} seems to be just a little greater than 1, we can conclude we are in the closed model. When λ is not used anymore to note the present value of \dot{a} , but rather its limit when $t \rightarrow +\infty$, our result is not the present value of Ω but its limit value. The present value have been observed to be, in the context of the standard cosmological model (Bennett and al., 2003) :

$$\Omega = 1.02 \pm 0.02 \quad (20.11)$$

To find $\Omega = 1.02$ in the quantum context, we need the present value of \dot{a} to be

$$\dot{a}_0 = \lambda = 7.07$$

and to find the greatest possibility $\Omega = 1.04$ we need

$$\dot{a}_0 = \lambda = 5$$

A remark can be made : if it can be observed, in our universe, distances of the order of $200Mpc$, and if the cH^{-1} distance is about $4000Mpc$, we then are sure that $\dot{a} \geq 1/20 = 0.05$. That the universe could be one hundred times bigger than this minimum value does not seem a priori to be ruled out by any experiment, and only very small values of \dot{a} are ruled out.

20.3 Value of one coefficient

We see that our equations have the remarkable property to explain, first, why the value of Ω is so near unity, but also that it is strictly greater than 1. The observed value of 1.02 fits perfectly with our equations, and proves furthermore that we are in the closed model. In the case where $\theta(a) = \theta_0 a^2$, we found the relation (17.5) for \dot{a} :

$$\dot{a} = \frac{1}{\sqrt{2\theta_0}} \sqrt{\ln(a/a_0)} \quad (20.12)$$

To give an approximate value of θ_0 , we can make the hypothesis that the approximate value of a_0 in the quantum theory is the value of the radius of the universe in the standard early phase of the universe. We find a relation of the kind

$$\frac{a}{a_0} \approx 10^{10} \quad (20.13)$$

Replacing this value in (20.11) we find $\dot{a} = 7.07$ for the value $\theta_0 \approx 5.3$, so the solution of the flatness and cosmological constant problems did not make appear a new θ_0 problem, since this time θ_0 is around unity.

21 Towards unification : the structure of matter and the ratio baryon to photon number

Analyzing the formula for p in the case $\theta(a) = \theta_0 a^2$, we find, from (15.11) :

$$p = \frac{\epsilon}{3} \left(1 - \frac{1}{\left(\frac{K}{a^2} + H^2\right) \theta(a)} \right) = \frac{\epsilon}{3} \left(1 - \frac{1}{K + \dot{a}^2 \theta_0} \right) \quad (21.1)$$

This is clear that this relation has not the right behavior, in order to predict $p \rightarrow 0$ when $t \rightarrow +\infty$, but it has the right form to predict once again a value of θ_0 near unity. Indeed, the condition $0 \leq p \leq \epsilon/3$ is equivalent to

$$\theta_0 \geq \frac{1}{K + \dot{a}^2}$$

The question here is to know what happens if the $p = 0$ condition cannot be verified anymore for the present universe. This delicate problem is studied in [57] and [58]. We just sketch here the ideas that will be developed there. There are two ways to get rid of the relation $p = 0$. The first is to consider a particle made of a small sphere of radius r , and huge energy density ϵ_P . Outside particles, in vacuum, there is no pressure, $p = 0$, but there is no energy density either : $\epsilon = 0$. Let us suppose now that the stuff making the particles is relativistic, in such a way that inside the particle we have the relation $p = \epsilon_P/3$. The values of p and ϵ which should be taken in the cosmological equation are the mean values of p and ϵ , these mean values being calculated over all parts of space, inside and outside particles. If, inside particles, we have $p = \epsilon_P/3$, outside particles this relation is still valid because there we have $p = \epsilon/3 = 0$. So the mean values of the pressure and the energy density still will verify the relation $p = \epsilon/3$ in the cosmological equation. As we can see, the value of p in the cosmological equation probes the structure of particles in the intermediate model. In the pointlike particles model, the relation between p and ϵ is calculated considering the mean relative velocities between particles. Now, if one electron for example, is made of more fundamental particles, which have great relative velocities between each other, but still are confined inside the electron, in the same manner that confined quarks live inside a proton, we find as well in the pointlike model a relation of the type $p = \epsilon/3$. Such hypotheses, which ultimately belong to the domain of application of unification, can be proved exact or at least very probable, because they fit with fundamental experimental observations as well as with the most theoretical pictures. From the theoretical point of view, if we imagine these more fundamental particles, which possess a relativistic speed, and a nonzero radius, and move, confined, inside the electron, we notice that these small spheres should be Lorentz contracted along the direction of their movement, in such a way that they should loose one dimension. Indeed the radius of the sphere along this direction should be $r = \sqrt{1 - v^2/c^2} \approx 0$, because we have $v \approx c$. So these more fundamental particles are membranes as in M-Theory. Such an hypothesis, like $p = \epsilon/3$, fits with observations, because the value of p does not only probes the structure of matter. It also gives information on the the total number N_B of baryons in the universe. A relation of the kind $p = \epsilon/3$ implies, since in the present universe

the energy density of photons is negligible, $p_B = \epsilon_B/3$, where p_B and ϵ_B are the baryonic pressure and energy density. Such a relation for baryons, associated with the conservation of entropy, and beyond their relativistic structure, gives a behavior of the baryonic energy density of the kind $\epsilon_B \sim 1/a^4$. It is well known that when the total number of baryons is constant, the number density is proportional to $1/a^3$. Here, since $\epsilon_B \sim 1/a^4$, we deduce that the total number of baryons is proportional to $1/a$. If the present value of the ratio η of the total number of baryons to the total number of photons has the observed value, in the context of general relativity, of $\eta \approx 6.1 \times 10^{10}$ (Bennett and al. 2003), and if the ratio a/a_0 of the present radius of the universe to the radius of the early universe has the value, computed with the standard cosmological model : $a/a_0 \approx 10^{10}$, we must conclude that the hypothesis $N_B \sim 1/a$ is just fine to obtain that in the early universe, there were about the same number of photons and baryons. Once again, such conclusions belong to the domain of unification, and will be treated at length in [56] and [57].

22 A limit case of the theory

There is a limit case of our quantum equation of gravity, which is the case $\theta \rightarrow +\infty$. Looking at the original equation of quantum gravity, we see that the term θ multiplies only the Gauss-Bonnet term. So if we multiply the two sides of the equation by $1/\theta$ and if we take the limit $\theta \rightarrow +\infty$, the theory we obtain in this limit implies the vanishing of the Gauss-Bonnet topological term. Furthermore, the equation for a , which took the form :

$$\frac{\ddot{a}}{a} = \frac{1}{4\theta(a)}$$

now becomes $\ddot{a} = 0$ or $\dot{a} = Cte$. We thus find a value of the deceleration parameter $q = 0$. The value of p is interesting since this time, from (15.9), we find the exact relativistic relation :

$$p = \epsilon/3$$

Also very interesting, in this limit, is the relation for ϵ :

$$\frac{K}{a^2} + H^2 = \frac{2}{3}\kappa_0\sqrt{\epsilon} \quad (22.1)$$

If we note $\dot{a}(t) = \lambda$, always possible since $\dot{a}(t)$ is constant, the relation (22.1) becomes :

$$\frac{\lambda'}{a^2} = \frac{2}{3}\kappa_0\sqrt{\epsilon} \quad (22.2)$$

where we have $\lambda' = K + \lambda^2$. We find that in this case, the closed, open and flat model are strictly equivalent, since they only differ by the value of λ' , which also depends on the initial condition λ , which is the value of \dot{a} at $t = 0$. So the flatness problem finds here a complete solution. The value of Ω_{TOT} is now constant and equal to :

$$\Omega_{TOT} = \left(1 + \frac{K}{\lambda^2}\right) \quad (22.3)$$

which is not necessarily equal to 1, but still, the values strictly greater than 1, strictly less than 1, or equal to 1 correspond respectively to the closed, open and flat model. The formula :

$$\Omega_\Lambda = \frac{1}{2}\Omega \quad (22.4)$$

which corresponds to the behavior

$$\kappa(\epsilon) \sim \epsilon^{-1/2}$$

remains the same for all values of θ . Thus, this relation is unchanged in the limit $\theta \rightarrow +\infty$. The origin of this Λ is what could be called a Gauss-Bonnet ghost term. It is a relic of the Gauss-Bonnet term, which does not vanish in this specific formula when we study the limit theory $\theta \rightarrow +\infty$, whereas the Gauss-Bonnet term itself vanishes in the equation of quantum gravity in this limit. Since in this case, Ω_{TOT} is constant, the term Ω_Λ is constant too, and the Λ term in the equation is strictly proportional to H^2 . We see that this limit case gives a perfect solution for the flatness problem as we said, but also a solution to the cosmological constant problem, via a Λ term having a ghost topological origin. This limit also gives a perfect relativistic structure for the fundamental particles, which corresponds to a solution to the problem corresponding to why the present value of the ratio η is so small. If we want to change the value Ω_Λ in (22.4) for the relation

$$\Omega_\Lambda = \frac{3}{4}\Omega \quad (22.5)$$

with the same ghost topological origin for Λ , it suffices to consider the case in which

$$\kappa(\epsilon) \sim \epsilon^{-3/4}$$

Indeed, relations (14.9) and (14.10) give

$$6 \left(\frac{K}{a^2} + H^2 \right) \left(\frac{1}{2} - \theta \frac{\ddot{a}}{a} \right) = \kappa(\epsilon)\epsilon \quad (22.6)$$

and

$$6 \left(\frac{K}{a^2} + H^2 \right) = 2 \int \kappa(\epsilon) d\epsilon \quad (22.7)$$

Taking now $\kappa(\epsilon) = k\epsilon^{-\alpha}$, we find :

$$6 \left(\frac{K}{a^2} + H^2 \right) = \frac{2k\epsilon^{1-\alpha}}{1-\alpha} \quad (22.8)$$

and we also find, using (22.6) :

$$\frac{2k}{1-\alpha} \left(\frac{1}{2} - \theta \frac{\ddot{a}}{a} \right) = k \quad (22.9)$$

We obtain :

$$\frac{\ddot{a}}{a} = \frac{\alpha}{2\theta} \quad (22.10)$$

We then use (14.4) and (14.5) :

$$-\Lambda = \theta \frac{\ddot{a}}{a} (b - 3c) \quad (22.11)$$

and (19.9) :

$$G_0^0 = \frac{(b - 3c)}{2} \quad (22.12)$$

so :

$$-\Lambda = \alpha G_0^0 \quad (22.13)$$

The conclusion is then :

$$\Omega_\Lambda = \alpha \Omega_{TOT} \quad (22.14)$$

So the value $\alpha = 3/4$ gives (22.5), but implies a gravitational coupling which has the dimension of an energy.

Part VI

Conservation of energy and topology

23 Introduction

23.1 Some well known facts

Conservation of energy If we look at the Einstein equations $R_{ik} - \frac{1}{2}Rg_{ik} = \kappa T_{ik}$ of general relativity, we see that the gravitational part is composed of a tensor G_{ik} , which verifies minimal conditions for the equation possible. The first condition, which enabled Einstein to find out his tensor, is that it should be constructed out of second derivatives of the fundamental variables of the theory, which are the g_{ik} . Mathematically, this means that G_{ik} must be constructed from the curvature tensor. Looking at the other side of the equation, we immediately see another necessary condition on G_{ik} , imposed by the law of conservation of energy $\nabla^i T_{ik} = 0$ on the matter tensor. So the equation is possible if and only if $\nabla^i G_{ik} = 0$. In fact, the tensor calculus provides us with this equation by a formal computation.

Dimension and topology However, the tensor $G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik}$ has dramatically different properties, depending on the dimension of space-time, and particularly its properties are different in the case $D = 2$ and when $D \geq 3$. In dimension $D = 2$, the Hilbert-Einstein action $\int \sqrt{-g}R$ is topological, and the Einstein tensor $R_{ik} - \frac{1}{2}Rg_{ik}$ possesses the condition of conformal invariance, we mean that its trace vanishes.

23.2 Constructing other tensors for gravity

A dimensionless coupling constant Using these first remarks, we consider the mathematical problem to construct all possible tensors Σ_{ik} , made of the curvature tensor, and

verifying the necessary law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. We will see that, if G_{ik} is the only tensor made of R_{ijkl} , of degree one in R_{ijkl} , and verifying the law of conservation of energy, there also is a unique tensor made of R_{ijkl} , of degree two in R_{ijkl} , and verifying the same law. The essential feature of this tensor is that possesses, in dimension $D = 4$, the properties of the Einstein tensor in $D = 2$. It is conformal invariant, we mean that its trace vanishes, and it has a dimensionless gravitational coupling constant. It is clear that it can be conjectured that there exists, for each integer n , a unique tensor made of R_{ijkl} , of degree n in R_{ijkl} , which is conformal invariant and which possesses a dimensionless coupling constant in dimension $D = 2n$

When topology appears In fact, these tensors of degree n in R_{ijkl} have, in their respective dimension $D = 2n$, another property of Einstein's tensor in $D = 2$: they are trivial, because they are topological. Thus, in dimension $D = 4$, starting from a tensor of degree 2 in R_{ijkl} , we can see in the calculation the following striking property : the sole condition of conservation of energy, makes appear in our tensor the exact coefficients of the topological Gauss-Bonnet term. In dimension $D = 2n$, the mathematical conjecture is that the sole equation of conservation of energy makes appear in the tensor of degree n in R_{ijkl} the coefficients of the Euler form. Since Donaldson invariants and then Seiberg-Witten invariants, we know a lot about the relations between physics and topology. Here, we use such a simple and direct relation between these two fields to construct another kind of quantum equation of gravity.

Complex gravity and the other quantum interactions In the quantum context, the wavy nature of matter is reflected by the fact, that in some way, complex field variables come into play. Looking carefully at a list in which all energy-momentum tensors, ready to quantization, are written down and put together, (Grib, Mamayev, Mostepanenko 1992, [18] Part I, Chapter 1), and by simple inspection, we observe general quantum features : all these tensors are of degree two in complex field variables and the doubling is made via complex conjugates. Applying the same rules, by analogy, to gravity, we arrive at a natural conclusion : gravity should be complex, we mean g_{ik} should be complex, the tensor

for gravity should be of degree 2, it thus should be the complex analog of the vanishing topological real tensor of degree 2, which makes appear $D = 4$ as a preferred dimension of space-time. We will not investigate more this complex tensor here, but if, in the complex case, this tensor is effectively non vanishing, we believe these links between reality and complexity, conservation of energy and topology, could be the key to understand why our world possesses four dimensions.

24 The tensor of degree two

24.1 Einstein's tensor of degree one

We just remember how we prove the existence and the uniqueness of G_{ik} of degree one in R_{ijkl} . As G_{ik} is of degree one in R_{ijkl} , only can it contain R_{ik} and R . So we have $G_{ik} = R_{ik} + \alpha R g_{ik}$ where α is a constant to be determined. Using the tensor calculus which gives formally $\nabla^i R_{ik} = \frac{1}{2} \partial_k R$, we see that $\nabla^i G_{ik} = (\alpha + \frac{1}{2}) \partial_k R = 0$ if and only if $\alpha = -\frac{1}{2}$. This gives the existence and the uniqueness of the tensor, as well as its exact expression. We now study the case of the degree two.

24.2 Ingredients for the tensor of degree two

The method Many more terms will contain the tensor of degree two, because in this case, there are the possibilities of using the four indexed R_{ijkl} , with indices contracted, as in $R_{iabc} R_k^{abc}$ or as in $R_{iakb} R^{ab}$. So we first have to determine all possible terms, and then calculate all the constants appearing in the linear combination forming our tensor. We recall that we note Σ_{ik} for this tensor. Next, using the law of conservation of energy for Σ_{ik} , we show that there is a unique solution to this set of constants.

The general form of the tensor To find the components of Σ_{ik} of degree two, we have simply to multiply two tensors of the form R_{abcd} , R_{ab} , or R and use as well g_{ik} , where the indices a, b, c, d are chosen to be i or k , or are otherwise contracted.

Products containing the scalar curvature R For a product R^2 the only possible term is $R^2 g_{ik}$, for a product of R_{ab} with R , again one possibility, which is RR_{ik} .

Products Ricci-Ricci For two products of R_{ab} , the indices i and k must belong to different R_{ab} , to avoid the appearance of the contraction R , a case already studied, and using that R_{ab} is symmetric, we get the only $R_{ia}R_k{}^a$.

Products Ricci-Riemann For products of R_{ab} and R_{abcd} , the term R_{ik} cannot appear, otherwise the contraction of R_{abcd} is R . As well, if R_{ia} appears, using the symmetries in the indices of R_{abcd} , we can suppose that k is the first index. We have then an expression of the form $R_{ia}R_{kpqr}$, where, among the indices, two are in the up position, one in the down position, a appears once, in the up position, to be contracted with the index a of R_{ia} , and say b appears twice among p, q and r , and is contracted. Then, in this Riemann tensor a cannot be the second index, otherwise the contraction over b is zero, so we can suppose a is the third index, and the contraction over b gives us another Ricci. So nor i neither k can appear in the Ricci, and we have then an R^{ab} where a, b are to be contracted with indices of a Riemann tensor. As R^{ab} is symmetric in a, b , it cannot be contracted with indices a, b placed in an antisymmetric position in R_{pqcd} , and as R_{pqcd} is antisymmetric in the first two indices and also in the last two, there is one a in the first two and one b in the last two. Using again the symmetries of the indices in the Riemann tensor, we can chose i in first place and k in the third, and we are left with the only possibility $R^{ab}R_{iakb}$.

Products Riemann-Riemann For the product of two Riemann tensors, it is quite direct to see that the only possibility is $R_i{}^{abc}R_{kabc}$. First, as before, we can suppose that i is the first index of the first Riemann. Now if i and k appear only in the first Riemann, c for example appears twice in the second, giving us zero or Ricci. So k is the first index of the second Riemann. Now we can chose the first as $R_i{}^{abc}$ and using the antisymmetry of the second tensor in the last two indices, we can suppose that in it, the last two indices are in alphabetical order. We are left with R_{kabc}, R_{kbac} and R_{kcab} . Using now that in the first Riemann b and c appear in antisymmetric positions, we have $R_i{}^{abc}R_{kbac} = -R_i{}^{abc}R_{kcab}$.

Using finally the identity $R_{kabc} - R_{kbac} + R_{kcab} = 0$, we see that all possible tensors can be written only in terms of $R_i^{abc} R_{kabc}$.

Terms involving the metric tensor In all this, we have discarded the possibility of the appearance of g_{ik} , but the same arguments permit to conclude that the only possible terms are $R^{(4)} g_{ik}$ where $R^{(4)} = R^{abcd} R_{abcd}$, $R^{(2)} g_{ik}$ where $R^{(2)} = R^{ab} R_{ab}$ and of course the $R^2 g_{ik}$ first considered.

Synthesis To conclude we get then the most general tensor Σ_{ik} of degree two :

$$\Sigma_{ik} = R_i^{abc} R_{kabc} + \alpha R_{iakb} R^{ab} + \beta R_{ia} R_k^a + \gamma R_{ik} R + (\delta R^{(4)} + \epsilon R^{(2)} + \eta R^2) g_{ik} \quad (24.1)$$

24.3 Three formulas

In order to calculate the coefficients appearing in $\nabla^i \Sigma_{ik}$, we need a first formula :

$$\nabla^i R_{iabc} = \nabla_b R_{ac} - \nabla_c R_{ab} \quad (24.2)$$

Starting with the Bianchi identity :

$$\nabla_m R_{nabc} + \nabla_c R_{namb} + \nabla_b R_{nacm} = 0 \quad (24.3)$$

and contracting over m and n, we obtain (24.2) directly :

$$\nabla^n R_{nabc} + \nabla_c R_{ab} - \nabla_b R_{ac} = 0$$

Here, we adopt the convention that the contraction of the first and the third indices in the Riemann tensor gives the Ricci tensor, and then the contraction of the first and fourth indices in the Riemann tensor gives minus the Ricci tensor, because of the antisymmetry of third and fourth indices in the Riemann tensor. This gives the formula (24.2). Now, we calculate the coefficients of $\nabla^i \Sigma_{ik}$ one by one. First we need a second formula :

$$\nabla^i (R_i^{abc} R_{kabc}) = (\nabla_b R_{ac}) R_k^{abc} - (\nabla_c R_{ab}) R_k^{abc} + 2R^{iabc} (\nabla_c R_{kabi}) \quad (24.4)$$

Using the properties of the connection ∇ , we find :

$$\nabla^i(R_i^{abc}R_{kabc}) = (\nabla^i R_i^{abc})R_{kabc} + R_i^{abc}(\nabla^i R_{kabc}) \quad (24.5)$$

and using

$$(\nabla^i R_i^{abc})R_{kabc} = (\nabla^i R_{iabc})R_k^{abc} \quad (24.6)$$

as well as equation (24.2), we obtain immediately that the first term of the right hand side of (24.5) equals the first two terms of formula (24.2). So, we only need to prove that the second term on the right hand side of (24.5) equals the third of formula (24.4). Now, using the Bianchi identity (24.3), we have :

$$R_i^{abc}(\nabla^i R_{kabc}) = -R_i^{abc}\nabla_b R_{kac}{}^i - R^{iabc}\nabla_c R_{kaib} \quad (24.7)$$

Using the antisymmetry of the indices b and c in the first Riemann tensor of the term $-R_i^{abc}\nabla_b R_{kac}{}^i$, we obtain that this term equals $R_i^{abc}\nabla_c R_{kab}{}^i$, which also equals the term $-R^{iabc}\nabla_c R_{kaib}$, using the antisymmetry of i and b in the second Riemann tensor. Altogether, we see that formula (24.4) has been proved. Now, we study the term arising in $\nabla^i \Sigma_{ik}$ from the second term of Σ_{ik} , asserting the following formula :

$$\nabla^i(\alpha R_{iakb}R^{ab}) = \alpha R^{ab}(\nabla_k R_{ab}) - \alpha R^{ab}(\nabla_b R_{ak}) - \alpha(\nabla_c R_{ab})R_k^{abc} \quad (24.8)$$

To prove it, we use :

$$\begin{aligned} \nabla^i(R_{iakb}R^{ab}) &= (\nabla^i R_{iakb})R^{ab} + R_{iakb}\nabla^i R^{ab} \\ &= (\nabla^i R_{iakb})R^{ab} + R^{ia}{}^b{}_k \nabla_i R_{ab} \end{aligned} \quad (24.9)$$

Now, applying to the first term on the right hand side of (24.9) the identity (24.2), we readily obtain the first two terms on the right hand side of formula (24.8). But the second term on the right hand side of (24.9) can be written $(\nabla_c R_{ab})R^{ca}{}^b{}_k$ and since exchanging the two pairs of indices in the Riemann tensor does not change its value, we obtain $(\nabla_c R_{ab})R_k{}^{bca}$. Now, using the well known identity

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0 \quad (24.10)$$

this term becomes

$$-(\nabla_c R_{ab})R_k^{abc} - (\nabla_c R_{ab})R_k^{cab}$$

In this last equation, the second term gives zero because a, b are contracted and appear in symmetric positions in the Ricci tensor and in antisymmetric positions in the Riemann tensor. This finishes the proof of formula (24.8).

24.4 Computing the coefficients of the tensor

Taking the β and γ terms of Σ_{ik} , and remembering that $\nabla^i R_{ik} = \frac{1}{2}\partial_k R$, we find at once :

$$\nabla^i(\beta R_{ia}R_k^a) = \frac{1}{2}\beta(\partial^a R)R_{ak} + \beta R^{ia}\nabla_i R_{ka} \quad (24.11)$$

and

$$\nabla^i(\gamma R_{ik}R) = \frac{1}{2}\gamma(\partial_k R)R + \gamma R_{ik}(\partial^i R) \quad (24.12)$$

Computing α, β, γ Looking closely at our equations, we see that the second terms of (24.8) and (24.11) can be eliminated by the choice $\alpha = \beta$, and that the first term of (24.11) gives zero, when combined with the second term of (24.12), provided that we choose the relation $\beta = -2\gamma$. So, by simple inspection of our equations, we possess an easy way to calculate our coefficients.

A relation which simplifies the whole calculation Turning now to the δ -term, we have :

$$\nabla^i (R^{(4)}g_{ik}) = \nabla^i [R_{abcd}R^{abcd}g_{ik}] = 2(\nabla_k R_{abcd})R^{abcd}$$

which equals :

$$-2R^{abcd}\nabla_d R_{abkc} - 2R^{abcd}\nabla_c R_{abdk}$$

because of (24.3). Both of these terms equal $2R^{abcd}\nabla_c R_{abkd}$, the second because in the second Riemann tensor, d and k are in antisymmetric positions, and the first because in the first Riemann tensor, c and d are in antisymmetric positions. We thus find :

$$\delta\nabla^i (R^{(4)}g_{ik}) = \delta\nabla^i [R_{abcd}R^{abcd}g_{ik}] = 4\delta R^{abcd}\nabla_c R_{abkd}$$

$$= 4\delta\nabla_c(R^{abcd}R_{abkd}) - 4\delta R_{abkd}\nabla_c R^{abcd}$$

Now happens a considerable simplification, because the first term of the former equation can be written $4\delta\nabla^c(R_{abcd}R_k{}^{ab}{}^d)$. Thus, c , which is contracted, can be called i , and we can exchange the two pairs of indices in both Riemann tensors, obtaining : $4\delta\nabla^i(R_{idab}R_k{}^{dab})$. This term is exactly the term of $\nabla^i\Sigma_{ik}$ which corresponds to the first term in the sum giving Σ_{ik} . So we find that choosing $\delta = -\frac{1}{4}$, we eliminate all the terms of (24.4). We are now left with a very few terms, the first and the third terms on the right hand side of (24.8), the first term on the right hand side of (24.12), the last $-4\delta R_{abkd}\nabla_c R^{abcd}$ and finally the ϵ and η terms of (24.1).

Computation of the δ -term This term can be written

$$-4\delta(\nabla^c R_{abcd})R_k{}^{ab}{}^d = (\nabla_a R_{db} - \nabla_b R_{ad})R_k{}^{ab}{}^d$$

using the value of δ and also formula (24.2). The second term on the right hand side of this formula is equal to the first, because in the Riemann tensor, a and b appear in antisymmetric positions, and we are left with :

$$2(\nabla_a R_{bd})R_k{}^{ab}{}^d = 2(\nabla_a R_{bd})R_k{}^{dab} = 2(\nabla_c R_{ab})R_k{}^{bca}$$

Indeed, we obtain the first equality by exchanging the pairs of indices in the Riemann tensor, and the second by renaming contracted indices. We use formula (24.10) to write $R_k{}^{bca} = -R_k{}^{abc} - R_k{}^{cab}$, and we observe that the second term has a and b in antisymmetric positions, which gives zero because these indices are contracted with $\nabla_c R_{ab}$. So the calculation of the δ -term of $\nabla^i\Sigma_{ik}$ is finished, and gives us only $-2(\nabla_c R_{ab})R_k{}^{abc}$, this term vanishing with the third term of (24.8) if and only if $\alpha = -2$. Comparing this result with the other relations obtained for β and γ , we now find $\beta = -2$, and $\gamma = +1$.

The ϵ -term We are now ready to study the ϵ -term. We know that we still have to cancel the first term of (24.8) and the first term of (24.12).

$$\epsilon\nabla^i[R_{ab}R^{ab}g_{ik}] = 2\epsilon(\nabla_k R_{ab})R^{ab}$$

cancels directly the first term of (24.8) provided $2\epsilon = -\alpha$, so $\epsilon = +1$.

The η -term The η -term gives $\eta \nabla^i [R^2 g_{ik}] = 2\eta R(\partial_k R)$, cancelling the first term of (24.12) provided $2\eta = -\frac{1}{2}\gamma$, which leads to $\eta = -\frac{1}{4}$, providing us finally a set of constants for which $\nabla^i \Sigma_{ik} = 0$. Finally we proved the statement of existence in the following theorem :

Theorem : There exists a unique tensor Σ_{ik} , constructed from all possible products of degree two of the Riemann tensor, its contractions, and the metric tensor, which verifies the law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. This tensor contains effectively all possible products and has the form :

$$\Sigma_{ik} = R_i^{abc} R_{kabc} - 2R_{iakb} R^{ab} - 2R_{ia} R_k^a + R_{ik} R - \frac{1}{4} (R^{(4)} - 4R^{(2)} + R^2) g_{ik} \quad (24.13)$$

where $R^{(4)} = R^{abcd} R_{abcd}$ and $R^{(2)} = R^{ab} R_{ab}$

Existence As we said, the existence in the theorem has been proved before, we just notice that we used for this proof all identities we know concerning the Riemann tensor and its contractions.

Uniqueness Of course, we have also proved that there was no more possible products which could be ingredients of the tensor Σ_{ik} , and that our coefficients formed the complete set of degrees of freedom of our mathematical problem. Finding these coefficients has been possible because we could cancel all terms in $\nabla^i \Sigma_{ik} = 0$, using the well known relations on the Riemann tensor. It appears that, as there does not exist any such other relation on this tensor, available in the generic situation, this was the unique manner of cancelling these terms, and that the coefficients of Σ_{ik} are unique. Here, we give a method to obtain an explicit proof of the uniqueness of Σ_{ik} : starting with the value of Σ_{ik} with all its coefficients, at first undetermined, we compute $\nabla^i \Sigma_{ik}$ in different explicit choices of the metric g_{ik} , and each example gives us a linear combination of our coefficients, that we put equal to zero. So we find a linear system for these coefficients and with enough choices of different g_{ik} , we obtain enough equations, to prove finally that only the coefficients of the theorem give zero in the generic situation.

25 Higher dimensions, topology and complex gravity

25.1 The conjecture in higher dimensions

From what has been done in the case of degree two, it is easily guessed what can be done as well in the case of degree n . We can consider a tensor Σ_{ik} , of degree n in the Riemann tensor and its contractions, and first find all possible products of degree n that could appear in Σ_{ik} . Then, we find all coefficients by imposing that in $\nabla^i \Sigma_{ik}$, all terms vanish. Looking at the case $n = 1$ and $n = 2$, it should be clear that it is a way of proving the following conjecture :

Conjecture : There is a unique tensor Σ_{ik} constructed from all possible products of degree n in the Riemann tensor and its contractions, constructed with the metric tensor too, and which verifies the law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. This tensor has the form :

$$\Sigma_{ik} = \tilde{\Sigma}_{ik} - \frac{1}{2n} \tilde{\Sigma} g_{ik}$$

where $\tilde{\Sigma}$ is the Euler form in dimension $2n$, as well as the trace of $\tilde{\Sigma}_{ik}$. Furthermore, Σ_{ik} vanishes, becomes it comes, using the calculus of variation, from the topological Euler lagrangian.

25.2 Topology

The appearance in the tensor of degree 2 of the Gauss-Bonnet term

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2$$

authorizes us to conjecture that our tensor completely vanishes in dimensions four, because it comes from the topological Gauss-Bonnet action :

$$\int \sqrt{-g} (R^{(4)} - 4R^{(2)} + R^2)$$

In dimensions different from four, the same tensor, of degree two, comes from the would-be-a-Gauss-Bonnet action :

$$\int \sqrt{-g}(R^{(4)} - 4R^{(2)} + R^2)$$

We thus have found an interesting method to write, from an a priori trivial topological action, a non trivial equation : start from the topological action in dimension n , go to another dimension where the same action is not trivial anymore, and use the calculus of variation to extract the tensorial equation. Then, take the tensor, and go back to the critical dimension. The question is : does the tensor obtained in this way should be discarded as being trivial or is it relevant to describe some kind of physics? This has been the first route which we used to find our equation of quantum gravity. Because the gravitational tensor of degree 2 first displays a dimensionless coupling constant, and second fits so well with the energy-momentum tensors of the other interactions, even if it is identically zero, we thought there should be some kind of physics behind. We finally retained the idea of keeping only its trace in the equation, which gave the Λ term, the law of conservation of energy being in the equation of quantum gravity being provided by the variations of $\kappa(\epsilon)$.

A dimensionless coupling constant Forgetting that our tensor vanishes for one moment, we consider the equation that such a tensor would give :

$$\Sigma_{ik} = \kappa T_{ik}$$

To determine the dimension of the coupling constant, we look at :

$$\Sigma_0^0 = \kappa T_0^0$$

Here the Riemann and Ricci tensors, when containing the same number of up and down indices, as well as the scalar curvature, have dimension $[L]^{-2}$, where $[L]$ is a length. So, Σ_0^0 has dimension $[L]^{-4}$. Now, T_0^0 equals ϵ , the energy density of matter, and has dimension, in dimension $D = 4$, $[E][L]^{-3} \sim \hbar[T]^{-1}[L]^{-3} \sim \hbar c[L]^{-4}$, since energy $[E]$ has dimension $\hbar[T]^{-1}$ and where of course $[T]$ is a time. Comparing these two results, we see that

$$\kappa = \frac{\kappa_0}{\hbar c}$$

where κ_0 has no dimension at all.

25.3 Complex gravity

We know that our tensor Σ_{ik} probably vanishes because it is the energy-momentum tensor coming from a topological lagrangian by the calculus of variations, but there is another form of this tensor, which at least at first sight, is not necessarily trivial, and which could prove itself very interesting. Because it is of second order in the curvature tensor, Σ_{ik} possesses a natural extension to complex gravity. As in the quantum tensors describing particles of different spin, we can write down a tensor of degree two by doubling the curvature terms by complex conjugates. By inspection of these known quantum tensors, we guess easily the procedure to follow. Indeed, we pose as new fundamental variables, the complex metric g_{ik} verifying the condition :

$$g_{ki}^* = g_{ik} \quad (25.1)$$

where z^* corresponds to the complex conjugate of z , and we pose the complex tensor :

$$\begin{aligned} \Sigma_{ik} = & \frac{1}{2} R^*_{iabc} R_{kabc} + \frac{1}{2} R_i{}^{abc} R^*_{kabc} - R^*_{iakb} R^{ab} - R_{iakb} R^{*ab} - R^*_{ia} R_k{}^a - R_{ia} R_k{}^{*a} \\ & + \frac{1}{2} R^*_{ik} R + \frac{1}{2} R_{ik} R^* - \frac{1}{4} (R^{(4)} - 4R^{(2)} + RR^*) g_{ik} \end{aligned} \quad (25.2)$$

where $R^{(4)} = R^*{}^{abcd} R_{abcd}$ and $R^{(2)} = R^*{}^{ab} R_{ab}$. Equations (25.1) and (25.2) should normally imply that Σ_{ik} is a real symmetric tensor which verifies the condition of conservation of energy.

Part VII

The Gauss-Bonnet term

26 Introduction

In Part VI, we have seen how the tensor Σ_{ik} , in the case of real gravity, which, independently of the fact that it vanishes, can be formally deduced from the conditions that it is of degree two in the Riemann tensor, and that it satisfies the law of conservation of energy $\nabla^i \Sigma_{ik} = 0$. We then saw that this tensor contains automatically in its trace the Gauss-Bonnet factor

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2 \quad (26.1)$$

where $R^{(4)} = R^{abcd}R_{abcd}$ and $R^{(2)} = R^{ab}R_{ab}$. The precise form of this tensor is :

$$\Sigma_{ik} = \tilde{\Sigma}_{ik} - \frac{1}{4}\tilde{\Sigma}g_{ik} \quad (26.2)$$

where :

$$\tilde{\Sigma}_{ik} = R_i{}^{abc}R_{kabc} - 2R_{iakb}R^{ab} - 2R_{ia}R_k{}^a + R_{ik}R \quad (26.3)$$

and where we note $\tilde{\Sigma}$ for the trace of $\tilde{\Sigma}_{ik}$. In this Part, we effectuate the complete computation of $\tilde{\Sigma}_{ik}$, in the case of the Robertson-Walker metric, in order to find $\tilde{\Sigma}$, an expression that we need because it appears in the quantum equation of gravity. In doing this, we will as a check verify the identity :

$$\Sigma_{ik} = 0 \quad (26.4)$$

27 The metric

27.1 Introduction

So we compute $\tilde{\Sigma}_{ik}$ in the case of the homogeneous and isotropic case, and for this we still take a metric of signature $(+1, -1, -1, -1)$ and indices going from 0 to 3. We note with greek indices $\alpha, \beta, \gamma, \delta \dots$ space indices going from 1 to 3. So we have

$$\eta_{00} = +1; \eta_{0\alpha} = 0; \eta_{\alpha\beta} = -\delta_{\alpha\beta} \quad (27.1)$$

where η_{ik} is Minkowski metric in four dimensions. We then derive the Robertson-Walker metric, following Landau, [33], paragraph 111, the computations of this derivation being necessary to compute all Riemann components, as it is necessary in order to obtain $\tilde{\Sigma}_{ik}$.

27.2 The spatial part of the calculation : the closed model

We first stick to the closed model, because there are simple relations to deduce the formulas of the open model from this particular case. Homogeneity and isotropy of space imply that the scalar curvature in three dimensions is constant, in the three-space variables. In fact, the Riemann tensor in three dimensions has enough symmetries to be computed :

$$P_{\alpha\beta\gamma\delta} = \lambda (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \quad (27.2)$$

and in the closed model, we choose this constant λ to be positive. To be very precise the metric tensor appearing in the last equation should be the euclidian metric tensor of the space of dimension three, which is the opposite of g_{ik} , because here the restriction of the signature of the space-time of dimension four is $-1, -1, -1$. Thus, the restriction of the metric of dimension four on the space of dimension three is the opposite of the euclidian metric in dimension three. However, as the components of $g_{\alpha\beta}$ appear multiplied in pairs in (27.2), this equation is still correct. We need to explain this in detail, because if this subtlety does not matter for the usual calculation of the Ricci tensor, it matters here a

lot, because it can make appear extra signs in $\tilde{\Sigma}_{ik}$, in case the calculation would not be done with care. Exactly, the three dimensional euclidian metric is

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} \quad (27.3)$$

and the Riemann tensor in three dimensions is :

$$P_{\alpha\beta\gamma\delta} = \lambda (\gamma_{\alpha\gamma}\gamma_{\beta\delta} - \gamma_{\alpha\delta}\gamma_{\beta\gamma}) \quad (27.4)$$

If we now take the Ricci tensor in three space by contracting this Riemann tensor, we obtain :

$$P_{\alpha\beta} = 2\lambda\gamma_{\alpha\beta} \quad (27.5)$$

This equation, as we said, makes appear an extra sign, when $g_{\alpha\beta}$ is used instead of $\gamma_{\alpha\beta}$, because then :

$$P_{\alpha\beta} = -2\lambda g_{\alpha\beta} \quad (27.6)$$

Finally, the scalar curvature is obtained and its value is :

$$P = 6\lambda \quad (27.7)$$

Now isotropy implies that $g_{0\alpha} = 0$ otherwise the vector field $g_{0\alpha} \neq 0$ would introduce by itself a space anisotropy. Imposing $g_{00} = 1$ means that we choose the time $t = \frac{x^0}{c}$ in our equations to be the physical time, that is to say the time showed by physical free falling clocks. We note $a(t)$ the inverse of the square root of the scalar curvature in three dimensions, which can be interpreted as the radius of the universe, we do not note R for this radius to avoid the confusion with the scalar curvature in four dimensions. Anyway we then have the relation :

$$\lambda = \frac{1}{a^2} \quad (27.8)$$

as can be seen for example in [33]. Now, using spherical coordinates in four dimensions, and choosing a frame which moves, at every point of space-time, with the physical free falling matter we find the value of ds^2 :

$$ds^2 = c^2 dt^2 - a^2(t)(d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (27.9)$$

where r, θ, ϕ are the variables of spherical coordinates in three dimensions and where $r = a(t) \sin \chi$, χ varying from 0 to π . Further, we can replace the time variable t by the dimensionless variable η , defined by $cdt = ad\eta$. We then obtains :

$$ds^2 = a^2(\eta)(d\eta^2 - d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)). \quad (27.10)$$

So we write our equations with variables x^0, x^1, x^2, x^3 being η, χ, ϕ, θ . We have from the previous equation :

$$g_{00} = a^2; g_{11} = -a^2; g_{22} = -a^2 \sin^2 \chi; g_{33} = -a^2 \sin^2 \chi \sin^2 \theta \quad (27.11)$$

and all non diagonal terms of g_{ik} vanish, such that the inverse matrix g^{ik} is straightforward.

27.3 Closed and open models

To go from the closed to the open model, we have to apply the following replacements : $a \rightarrow ia$; $\eta \rightarrow i\eta$; $\chi \rightarrow i\chi$. Finally $t \rightarrow -t$ can be deduced from the former rules, and from the relation $cdt = ad\eta$. As an example, the metric for the closed model transforms, in the case of the open model, into :

$$ds^2 = a^2(\eta)(d\eta^2 - d\chi^2 - \sinh^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (27.12)$$

In detail, we see that $t \rightarrow -t$ and dt^2 is invariant. $a^2 \rightarrow -a^2$, $d\chi^2 \rightarrow -d\chi^2$, and finally $\sin^2 \chi \rightarrow -\sinh^2 \chi$, which establishes the form of the metric in the open case, from the metric in the closed case. As for λ , the relation $\lambda = 1/a^2$ transforms to

$$\lambda = \frac{K}{a^2} \quad (27.13)$$

where K takes the values : $K = +1$ in the closed case, and $K = -1$ in the open case.

27.4 The Christoffel symbols

We first stick to the closed model, at the end of the calculation we shall deduce, from these results, the formulas for the open model. We use primes to note the η -derivation

and dots to note the t -derivation. We use the general formula :

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \quad (27.14)$$

We recall here that latin indices are four dimensional indices and greek indices are three dimensional ones. In the case of Einstein's equations, the computation is easier because one only need the Ricci tensor. To compute $\tilde{\Sigma}_{ik}$, we need all values of the Riemann tensor, and we do have to compute the values of Γ_{jk}^i with care. What makes the computation easier, is that both g_{ik} and g^{ik} are diagonal. We compute :

$$\Gamma_{00}^0 = \frac{1}{2}g^{00}(\partial_0 g_{00}) = \frac{1}{2}\frac{1}{a^2}2aa' = \frac{a'}{a} \quad (27.15)$$

and

$$\Gamma_{\alpha\beta}^0 = \frac{1}{2}\frac{1}{a^2}(\partial_\alpha g_{0\beta} + \partial_\beta g_{0\alpha} - \partial_0 g_{\alpha\beta}) = -\frac{a'}{a^3}g_{\alpha\beta} \quad (27.16)$$

We further have :

$$\Gamma_{0\beta}^\alpha = \frac{1}{2}g^{\alpha\alpha}(\partial_0 g_{\alpha\beta} + \partial_\beta g_{0\alpha} - \partial_\alpha g_{0\beta}) = \frac{1}{2}g^{\alpha\alpha}(\partial_0 g_{\alpha\beta}) = \frac{1}{2}g^{\alpha\alpha}\frac{2a'}{a}g_{\alpha\beta} = \frac{a'}{a}\delta_\beta^\alpha \quad (27.17)$$

As well can we see that :

$$\Gamma_{0\alpha}^0 = \Gamma_{00}^\alpha = 0 \quad (27.18)$$

We will see that we do not need the components $\Gamma_{\beta\gamma}^\alpha$.

28 The Riemann tensor

Next, we compute all Riemann tensor components, using the classical formula :

$$R_{iklm} = \frac{1}{2} [\partial_{kl}^2 g_{im} + \partial_{im}^2 g_{kl} - \partial_{il}^2 g_{km} - \partial_{km}^2 g_{il}] + g_{np} (\Gamma_{kl}^n \Gamma_{im}^p - \Gamma_{km}^n \Gamma_{il}^p) \quad (28.1)$$

We separate cases, according to the number of space indices a component of a tensor possesses. For example, we say that $R_{\alpha\beta\gamma\delta}$ is a four space indices component, since it does not possess any time index. As another example, $R_{\alpha 0\beta\gamma}$ is named a three space indices component. We need to precise that the indices are only counted when they all are down.

28.1 Riemann : four space indices

Now, if we compute $R_{\alpha\beta\gamma\delta}$ using the previous formula, we first find that the Riemann tensor contains only odd products of the metric tensor, and changing the metric into its opposite transforms the Riemann tensor into its opposite. Second, the previous formula contains a sum over indices p and n , which means, for both indices, a sum over the three space indices and also over the 0 time index. If we put together all terms from the summation over n and p , only when n and p vary over space indices, as well as the first four terms of equation (28.1), we get minus the Riemann tensor in three dimensions. Concerning the sum over p or n when one of them equals 0, taken into account that $g_{0\alpha} = 0$, we only obtain the term corresponding to the case $n = p = 0$:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -P_{\alpha\beta\gamma\delta} + g_{00}(\Gamma_{\beta\gamma}^0 \Gamma_{\alpha\delta}^0 - \Gamma_{\beta\delta}^0 \Gamma_{\alpha\gamma}^0) \\ &= -\frac{1}{a^2} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) + a^2 \left[\left(\left(-\frac{a'}{a^3} \right) g_{\beta\gamma} \left(-\frac{a'}{a^3} \right) g_{\alpha\delta} \right) - \left(\left(-\frac{a'}{a^3} \right) g_{\beta\delta} \left(-\frac{a'}{a^3} \right) g_{\alpha\gamma} \right) \right] \end{aligned}$$

So finally :

$$R_{\alpha\beta\gamma\delta} = \frac{a^2 + a'^2}{a^4} (g_{\beta\gamma} g_{\alpha\delta} - g_{\beta\delta} g_{\alpha\gamma}) \quad (28.2)$$

28.2 Riemann : three space indices

Now we compute $R_{\alpha\beta\gamma 0}$: using (28.1) and the fact the metric tensor is diagonal we find :

$$R_{\alpha\beta\gamma 0} = \frac{1}{2} (\partial_{\alpha 0}^2 g_{\beta\gamma} - \partial_{\beta 0}^2 g_{\alpha\gamma}) + g_{np} (\Gamma_{\beta\gamma}^n \Gamma_{\alpha 0}^p - \Gamma_{\beta 0}^n \Gamma_{\alpha\gamma}^p)$$

If we try to sum the terms over n and p , we see that the term corresponding to $n = p = 0$ contains only products containing one $\Gamma_{\alpha 0}^0 = 0$, and thus vanishes. We are left with a sum over λ and μ :

$$g_{\lambda\mu} (\Gamma_{\beta\gamma}^{\lambda} \Gamma_{\alpha 0}^{\mu} - \Gamma_{\beta 0}^{\lambda} \Gamma_{\alpha\gamma}^{\mu}) = \frac{a'}{a} (g_{\lambda\alpha} \Gamma_{\beta\gamma}^{\lambda} - g_{\beta\mu} \Gamma_{\alpha\gamma}^{\mu}) = \frac{a'}{a} (\Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma})$$

the first equality coming from

$$\Gamma_{\alpha 0}^{\mu} = \frac{a'}{a} \delta_{\alpha}^{\mu}$$

So we can use, because $g_{\alpha 0} = 0$:

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha n} \Gamma_{\beta\gamma}^n = g_{\alpha\lambda} \Gamma_{\beta\gamma}^\lambda = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma})$$

and collecting all terms, without forgetting the two derivatives of the metric tensor :

$$R_{\alpha\beta\gamma 0} = \frac{1}{2} (\partial_{\alpha 0}^2 g_{\beta\gamma} - \partial_{\beta 0}^2 g_{\gamma\delta}) + \frac{a'}{a} (\partial_\beta g_{\gamma\delta} - \partial_\alpha g_{\beta\gamma})$$

We notice that if we write $g_{\alpha\beta} = a^2 \tilde{g}_{\alpha\beta}$, then $\tilde{g}_{\alpha\beta}$ does not depend on $\eta = x^0$, and we can write

$$\partial_0 g_{\alpha\gamma} = 2 \frac{a'}{a} g_{\alpha\gamma}$$

We thus obtain

$$\frac{1}{2} \partial_{\alpha 0}^2 g_{\beta\gamma} = \frac{a'}{a} \partial_\alpha g_{\beta\gamma}$$

because $a = a(\eta)$ does not depend on any space variable. We finally obtain :

$$R_{\alpha\beta\gamma 0} = 0 \tag{28.3}$$

28.3 Riemann : two space indices

We have now to compute $R_{\alpha 0 \beta 0}$, which actually are the last components of the Riemann tensor which may not vanish. Indeed, any component of the Riemann tensor containing three or more indices equal to zero vanishes, because of the antisymmetry of the two first indices, and also because of the antisymmetry of the two last indices. Using again the general formula (28.1) for the Riemann tensor, we write :

$$R_{\alpha 0 \beta 0} = \frac{1}{2} (-\partial_{00}^2 g_{\alpha\beta} - \partial_{\alpha\beta}^2 g_{00}) + g_{np} (\Gamma_{0\beta}^n \Gamma_{\alpha 0}^p - \Gamma_{00}^n \Gamma_{\alpha\beta}^p)$$

and since

$$\partial_0 g_{\alpha\beta} = 2 \frac{a'}{a} g_{\alpha\beta}$$

we compute :

$$\partial_{00}^2 g_{\alpha\beta} = \left[\left(2 \frac{\dot{a}}{a} \right)' + 4 \frac{a'^2}{a^2} \right] g_{\alpha\beta} = 2 \left(\frac{\ddot{a}a - a'^2}{a^2} + 2 \frac{a'^2}{a^2} \right) g_{\alpha\beta}$$

$$= 2 \left(\frac{aa'' + a'^2}{a^2} \right) g_{\alpha\beta}$$

We also have :

$$\partial_{\alpha\beta}^2 g_{00} = 0$$

because $\partial_\alpha a = 0$, since $a = a(\eta)$ does not depend on any space variable. The first term of

$$g_{00}(\Gamma_{0\beta}^0 \Gamma_{\alpha 0}^0 - \Gamma_{00}^0 \Gamma_{\alpha\beta}^0)$$

vanishes, because $\Gamma_{\alpha 0}^0 = 0$. Using the values of Γ_{jk}^i , we find that its second term, taking into account the minus sign, equals :

$$\left(\frac{a'}{a} \right)^2 g_{\alpha\beta}$$

Finally, the second term of :

$$g_{\lambda\mu} (\Gamma_{0\beta}^\lambda \Gamma_{\alpha 0}^\mu - \Gamma_{00}^\lambda \Gamma_{\alpha\beta}^\mu)$$

vanishes, because $\Gamma_{00}^\lambda = 0$, whereas its first term equals :

$$g_{\lambda\mu} \left(\frac{a'}{a} \right) \delta_\beta^\lambda \left(\frac{a'}{a} \right) \delta_\alpha^\mu = \left(\frac{a'}{a} \right)^2 g_{\alpha\beta}$$

Collecting all terms, we obtain :

$$R_{\alpha 0 \beta 0} = \frac{a'^2 - aa''}{a^2} g_{\alpha\beta} \quad (28.4)$$

The Riemann tensor completely computed, we can now contract indices to find the Ricci tensor, the scalar curvature, and finally Σ_{ik} .

29 The Ricci tensor

29.1 Two time indices

We recall that we count indices in tensors when they are all down, according to the exact number of time indices 0 which appear, or equivalently according to the exact number of

space indices which appear. For the Ricci tensor, there is only one component with two time indices, which is R_{00} . From

$$R^\alpha{}_{0\beta 0} = \left(\frac{a'^2 - aa''}{a^2} \right) \delta_\beta^\alpha$$

we compute

$$R_{00} = R^\alpha{}_{0\alpha 0} = \left(\frac{a'^2 - aa''}{a^2} \right) \delta_\alpha^\alpha = 3 \left(\frac{a'^2 - aa''}{a^2} \right) \quad (29.1)$$

and

$$R_0^0 = g^{00} R_{00} = \frac{3(a'^2 - aa'')}{a^4} = b \quad (29.2)$$

which defines b .

29.2 One space index

We notice that there is no component of the Riemann tensor containing an odd number of indices equal to 0, that is to say time indices. We thus compute :

$$R_{\alpha 0} = R^n{}_{\alpha n 0} = R^\gamma{}_{\alpha \gamma 0} + R^0{}_{\alpha 0 0} = 0 \quad (29.3)$$

29.3 Two space indices and scalar curvature

Finally, from

$$R^\alpha{}_{\beta\gamma\delta} = \left(\frac{a^2 + a'^2}{a^4} \right) (\delta_\delta^\alpha g_{\beta\gamma} - \delta_\gamma^\alpha g_{\beta\delta})$$

we find

$$\begin{aligned} R_{\beta\delta} &= R^n{}_{\beta n \delta} = R^\gamma{}_{\beta\gamma\delta} + R^0{}_{\beta 0 \delta} = \left(\frac{a^2 + a'^2}{a^4} \right) (g_{\beta\delta} - 3g_{\beta\delta}) + g^{00} R_{\beta 0 \delta 0} \\ &= \left(-\frac{2}{a^4} (a^2 + a'^2) + \frac{1}{a^4} (a'^2 - aa'') \right) g_{\beta\gamma} \end{aligned}$$

We find :

$$R_{\beta\delta} = -\frac{1}{a^4} (2a^2 + a'^2 + aa'') g_{\beta\delta} = c g_{\beta\delta} \quad (29.4)$$

which defines c . We then have the scalar curvature :

$$R = b + 3c = -\frac{6}{a^3}(a + a^n) = d \quad (29.5)$$

which defines d . We are ready now to compute the tensor Σ_{ik} using all our previous calculations.

30 The Gauss-Bonnet tensor

30.1 Ricci-times-scalar curvature and Ricci-times-Ricci

We start with $R_{ik}R$ and $R_{ia}R_k^a$. From equations $g_{00} = a^2$ of (27.11), from (29.1), (29.2), (29.4) and (29.5), we obtain :

$$R_{\alpha\beta}R = cdg_{\alpha\beta} = c(b + 3c)g_{\alpha\beta} \quad (30.1)$$

and

$$R_{00}R = bdg_{00} = b(b + 3c)g_{00} \quad (30.2)$$

and also from (29.3) :

$$R_{\alpha 0}R = 0 \quad (30.3)$$

We have from (29.3), $R_{\alpha 0} = 0$:

$$R_{\alpha a}R_\beta^a = R_{\alpha\gamma}R_\beta^\gamma + R_{\alpha 0}R_\beta^0 = c^2g_{\alpha\gamma}\delta_\beta^\gamma = c^2g_{\alpha\beta} \quad (30.4)$$

and

$$R_{\alpha a}R_0^a = R_{\alpha 0}R_0^0 + R_{\alpha\gamma}R_0^\gamma = 0 \quad (30.5)$$

We also have :

$$R_{0a}R_0^a = R_{00}R_0^0 + R_{0\gamma}R_0^\gamma = b^2g_{00} \quad (30.6)$$

30.2 Riemann-times-Ricci

We compute then $R_{iakb}R^{ab}$. In $R_{\alpha a 0 b}R^{ab}$, we know that $R^{ab} = 0$ if an odd number of the indices among a and b are 0, and $R_{\alpha a 0 b}$ is 0 if an odd number of indices among $\alpha, a, 0$ and b are zero, which means a even number of indices among a and b are 0. So all terms cancel, and :

$$R_{\alpha a 0 b}R^{ab} = 0 \quad (30.7)$$

Furthermore, we have :

$$R_{\alpha a \beta b}R^{ab} = R_{\alpha 0 \beta 0}R^{00} + R_{\alpha \gamma \beta \delta}R^{\gamma \delta} = bR_{\alpha 0 \beta}{}^0 + cR_{\alpha \gamma \beta}{}^\gamma \quad (30.8)$$

The first equality results from (29.3), and the second from (29.2), (29.3) and (29.4), and also because :

$$R_{\alpha \gamma \beta \delta}R^{\gamma \delta} = cR_{\alpha \gamma \beta \delta}g^{\gamma \delta} = R_{\alpha \gamma \beta n}g^{\gamma n}$$

since $g^{\gamma 0} = 0$. Now, we have :

$$R_{\alpha \gamma \beta}{}^\gamma = R_{\alpha a \beta}{}^a - R_{\alpha 0 \beta}{}^0 = R_{\alpha \beta} - R_{\alpha 0 \beta}{}^0$$

so :

$$R_{\alpha a \beta b}R^{ab} = (b - c)g^{00}R_{\alpha 0 \beta 0} + cR_{\alpha \beta}$$

We find from (28.4) and (29.2) that :

$$R_{\alpha 0 \beta 0} = \frac{ba^2}{3}g_{\alpha \beta} \quad (30.9)$$

We also have $g^{00} = \frac{1}{a^2}$, and using again (29.4) :

$$R_{\alpha a \beta b}R^{ab} = \frac{b(b - c)}{3}g_{\alpha \beta} + c^2g_{\alpha \beta} = \frac{b^2 - bc + 3c^2}{3}g_{\alpha \beta} \quad (30.10)$$

Now, using (29.3) :

$$R_{0 a 0 b}R^{ab} = R_{0000}R^{00} + R_{0 \alpha 0 \beta}R^{\alpha \beta}$$

However, $R_{0000} = 0$, since it possesses its first two indices in antisymmetric positions and equal, so :

$$R_{0 a 0 b}R^{ab} = R_{0 \alpha 0 \beta}R^{\alpha \beta} = cg^{\alpha \beta}R_{0 \alpha 0 \beta} = cg^{ab}R_{0 a 0 b} = cg^{ab}R_{a 0 b 0}$$

since $g^{\alpha 0} = g^{0\beta} = 0$, and $R_{0000} = 0$. Finally, we obtain :

$$R_{0a0b}R^{ab} = cR_{00} = bcg_{00} \quad (30.11)$$

from (29.2).

30.3 Riemann-times-Riemann

We need also to compute $R_i^{abc}R_{kabc}$.

We first have :

$$R_\alpha^{abc}R_{0abc} = 0 \quad (30.12)$$

since the first term in the product vanishes for an odd number of 0 among the indices a, b and c , whereas the second term vanishes, for an even number of them. In $R_0^{abc}R_{0abc}$, we have $a \neq 0$ otherwise $R_{0abc} = 0$, and we can put $a = \alpha$. We recall that $R_{ijkl} = 0$ for an odd number of 0 among the indices i, j, k and l . Thus, exactly one index between b and c is 0, if we impose to R_{0abc} being nonzero.

$$R_0^{abc}R_{0abc} = R_0^{\alpha bc}R_{0abc} = R_0^{\alpha 0\gamma}R_{0\alpha 0\gamma} + R_0^{\alpha\beta 0}R_{0\alpha\beta 0}$$

and since the last two indices in R_{ijkl} are antisymmetric :

$$R_0^{\alpha bc}R_{0abc} = 2R_0^{\alpha 0\gamma}R_{0\alpha 0\gamma} = \frac{2b}{3}a^2g_{\alpha\gamma}R_0^{\alpha 0\gamma} = \frac{2b}{3}a^2R_0^0$$

because

$$R_{0\alpha 0\gamma} = R_{\alpha 0\gamma 0} = \frac{a'^2 - aa''}{a^2}g_{\alpha\gamma} = \frac{ba^2}{3}g_{\alpha\gamma}$$

Finally :

$$R_0^{abc}R_{0abc} = \frac{2b^2}{3}a^2 = \frac{2b^2}{3}g_{00} \quad (30.13)$$

Here, we recall that $g_{00} = a^2$. We also have to evaluate $R_\alpha^{abc}R_{\beta abc}$. In this expression, when it does not vanish, in each term, the number of 0 among a, b and c is even. So, this number can only be 0 or 2. Furthermore b and c cannot be equal, so cannot be 0 at the same time. In these conditions, when the number of 0 between a, b and c is 2, we must have $a = 0$. Because of this, we obtain :

$$R_\alpha^{abc}R_{\beta abc} = R_\alpha^{\gamma\delta\epsilon}R_{\beta\gamma\delta\epsilon} + R_\alpha^{00\gamma}R_{\beta 00\gamma} + R_\alpha^{0\gamma 0}R_{\beta 0\gamma 0}$$

$$= R_{\alpha}{}^{\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon} + 2R_{\alpha}{}^{0\gamma 0} R_{\beta 0\gamma 0} \quad (30.14)$$

From (30.9) :

$$\begin{aligned} 2R_{\alpha}{}^{0\gamma 0} R_{\beta 0\gamma 0} &= \frac{2b}{3} a^2 g_{\beta\gamma} R_{\alpha}{}^{0\gamma 0} = \frac{2b}{3} g_{00} R_{\alpha}{}^0{}_{\beta}{}^0 \\ &= \frac{2b}{3} R_{\alpha 0\beta}{}^0 = \frac{2b}{3} g^{00} R_{\alpha 0\beta 0} = \frac{2b^2}{9} g_{\alpha\beta} \end{aligned} \quad (30.15)$$

from (30.9) again. Now we evaluate $R_{\alpha}{}^{\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon}$. We have already computed the term :

$$R_{\alpha\gamma\delta\epsilon} = \tilde{\lambda} (g_{\gamma\delta} g_{\alpha\epsilon} - g_{\gamma\epsilon} g_{\alpha\delta})$$

where

$$\tilde{\lambda} = \frac{a^2 + a'^2}{a^4}$$

We find :

$$R_{\alpha}{}^{\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon} = \tilde{\lambda}^2 (g_{\gamma\delta} g_{\beta\epsilon} - g_{\gamma\epsilon} g_{\beta\delta}) (g^{\gamma\delta} \delta_{\alpha}^{\epsilon} - g^{\gamma\epsilon} \delta_{\alpha}^{\delta}) = 4\tilde{\lambda}^2 g_{\alpha\beta} \quad (30.16)$$

Indeed, we have :

$$g_{\gamma\delta} g_{\beta\epsilon} g^{\gamma\delta} \delta_{\alpha}^{\epsilon} = g_{\gamma\delta} g^{\gamma\delta} g_{\alpha\beta} = \delta_{\gamma}^{\gamma} g_{\alpha\beta} = 3g_{\alpha\beta}$$

and

$$-g_{\gamma\delta} g_{\beta\epsilon} g^{\gamma\epsilon} \delta_{\alpha}^{\delta} = -g_{\gamma\alpha} g_{\beta\epsilon} g^{\gamma\epsilon} = -g_{\gamma\alpha} \delta_{\beta}^{\delta} = -g_{\beta\alpha} = -g_{\alpha\beta}$$

We know that

$$\frac{b}{3} = \frac{a'^2 - aa''}{a^4}$$

and also

$$c = -\frac{2a^2 + a'^2 + aa''}{a^4}$$

We thus can compute :

$$\frac{b}{3} - c = \frac{2a^2 + 2a'^2}{a^4} \quad (30.17)$$

We finally obtain the value of $\tilde{\lambda}$:

$$\tilde{\lambda} = \frac{b - 3c}{6} \quad (30.18)$$

Putting together all these results, we arrive at :

$$R_{\alpha}{}^{abc} R_{\beta abc} = \left(\frac{2b^2}{9} + 4\tilde{\lambda}^2 \right) g_{\alpha\beta} = \left(\frac{2b^2}{9} + \frac{1}{9}(b - 3c)^2 \right) g_{\alpha\beta}$$

so we obtain :

$$R_{\alpha}{}^{abc} R_{\beta abc} = \left(\frac{b^2 - 2bc + 3c^2}{3} \right) g_{\alpha\beta} \quad (30.19)$$

30.4 The Gauss-Bonnet tensor

We write (26.3) :

$$\tilde{\Sigma}_{ik} = R_i^{abc} R_{kabc} - 2R_{iakb} R^{ab} - 2R_{ia} R_k^a + R_{ik} R$$

as well as (26.1):

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2$$

where $\tilde{\Sigma}$ is the trace of $\tilde{\Sigma}_{ik}$ and we have (26.2):

$$\Sigma_{ik} = \tilde{\Sigma}_{ik} - \frac{1}{4} \tilde{\Sigma} g_{ik}$$

So picking up the terms in (30.3),(30.5),(30.7) and (30.12) we obtain :

$$\tilde{\Sigma}_{\alpha 0} = 0 \tag{30.20}$$

Picking up the terms in (30.2), (30.6), (30.11) and (30.13), we obtain :

$$\tilde{\Sigma}_{00} = \left(\frac{2b^2}{3} - 2bc - 2b^2 + b(b + 3c) \right) g_{00} = \frac{b}{3}(3c - b)g_{00} \tag{30.21}$$

Picking up the terms in (30.1), (30.4), (30.10) and (30.19), we obtain :

$$\tilde{\Sigma}_{\alpha\beta} = \left[\left(\frac{b^2 - 2bc + 3c^2}{3} \right) - 2 \left(\frac{b^2 - bc + 3c^2}{3} \right) - 2c^2 + c(b + 3c) \right] g_{\alpha\beta}$$

So

$$\tilde{\Sigma}_{\alpha\beta} = \frac{b}{3}(3c - b)g_{\alpha\beta} \tag{30.22}$$

And finally, we see that $\tilde{\Sigma}_{ik}$ is diagonal. Thus Σ_{ik} also is diagonal, being at the same time of vanishing trace. So $\Sigma_{ik} = 0$, as a check of all the computations of this part.

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