Conservation of energy and Gauss Bonnet gravity

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I dedicate this work to Odilia.

Abstract

In the present article, we prove that any tensor of degree two in the Riemann tensor, which satisfies the principle of conservation of energy, vanishes identically, in dimension $D = 4$, because it comes from the topological Gauss-Bonnet lagrangian. We conjecture that, in dimension $D = 2n$, the topological Euler form can also be deduced from the conservation of energy. We then give arguments in favor of complex gravity, and write a tensor that is likely to describe quantum gravity, with a dimensionless gravitational coupling constant.

1 Introduction

1.1 Some well known facts

Conservation of energy   If we look at the Einstein equations $R_{ik} - \frac{1}{2}Rg_{ik} = \kappa T_{ik}$ of general relativity, we see that the gravitational part is composed of a tensor G_{ik}, which

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verifies minimal conditions for the equation possible. The first condition, which enabled Einstein to find out his tensor, is that it should be constructed out of second derivatives of the fundamental variables of the theory, which are the g_{ik}. Mathematically, this means that G_{ik} must be constructed from the curvature tensor. Looking at the other side of the equation, we immediately see another necessary condition on G_{ik}, imposed by the law of conservation of energy $\nabla^i T_{ik} = 0$ on the matter tensor. So the equation is possible if and only if $\nabla^i G_{ik} = 0$. In fact, the tensor calculus provides us with this equation by a formal computation.

Dimension and topology However, the tensor $G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik}$ has dramatically different properties, depending on the dimension of space-time, and particularly its properties are different in the case $D = 2$ and when $D \geq 3$. In dimension $D = 2$, the Hilbert-Einstein action $\int \sqrt{-g}R$ is topological, and the Einstein tensor $R_{ik} - \frac{1}{2}Rg_{ik}$ possesses the condition of conformal invariance, we mean that its trace vanishes.

1.2   Constructing other tensors for gravity

A dimensionless coupling constant Using these first remarks, we consider the mathematical problem to construct all possible tensors Σ_{ik}, made of the curvature tensor, and verifying the necessary law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. We will see that, if G_{ik} is the only tensor made of R_{ijkl}, of degree one in R_{ijkl}, and verifying the law of conservation of energy, there also is a unique tensor made of R_{ijkl}, of degree two in R_{ijkl}, and verifying the same law. The essential feature of this tensor is that possesses, in dimension $D = 4$, the properties of the Einstein tensor in $D = 2$. It is conformal invariant, we mean that its trace vanishes, and it has a dimensionless gravitational coupling constant. It is clear that it can conjectured that there exists, for each integer n, a unique tensor made of R_{ijkl}, of degree n in R_{ijkl}, which is conformal invariant and which possesses a dimensionless coupling constant in dimension $D = 2n$

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When topology appears In fact, these tensors of degree n in R_{ijkl} have, in their respective dimension $D = 2n$, another property of Einstein's tensor in $D = 2$: they are trivial, because they are topological. Thus, in dimension $D = 4$, starting from a tensor of degree 2 in R_{ijkl}, we can see in the calculation the following striking property : the sole condition of conservation of energy, makes appear in our tensor the exact coefficients of the topological Gauss-Bonnet term. In dimension $D = 2n$, the mathematical conjecture is that the sole equation of conservation of energy makes appear in the tensor of degree n in R_{ijkl} the coefficients of the Euler form. Since Donaldson invariants and then Seiberg-Witten invariants, we know a lot about the relations between physics and topology. Here, we use such a simple and direct relation between these two fields to construct another kind of quantum equation of gravity.

Complex gravity and the other quantum interactions In the quantum context, the wavy nature of matter is reflected by the fact, that in some way, complex field variables come into play. Looking carefully at a list in which all energy-momentum tensors, ready to quantization, are written down and put together, (Grib, Mamayev, Mostepanenko 1992, [**?**] Part I, Chapter 1), and by simple inspection, we observe general quantum features : all these tensors are of degree two in complex field variables and the doubling is made via complex conjugates. Applying the same rules, by analogy, to gravity, we arrive at a natural conclusion : gravity should be complex, we mean g_{ik} should be complex, the tensor for gravity should be of degree 2, it thus should be the complex analog of the vanishing topological real tensor of degree 2, which makes appear $D = 4$ as a preferred dimension of space-time. We will not investigate more this complex tensor here, but if, in the complex case, this tensor is effectively non vanishing, we believe these links between reality and complexity, conservation of energy and topology, could be the key to understand why our world possesses four dimensions.

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2 The tensor of degree two

2.1 Einstein's tensor of degree one

We just remember how we prove the existence and the uniqueness of G_{ik} of degree one in R_{ijkl}. As G_{ik} is of degree one in R_{ijkl}, only can it contain R_{ik} and R. So we have $G_{ik} = R_{ik} + \alpha R g_{ik}$ where α is a constant to be determined. Using the tensor calculus which gives formally $\nabla^i R_{ik} = \frac{1}{2} \partial_k R$, we see that $\nabla^i G_{ik} = (\alpha + \frac{1}{2}) \partial_k R = 0$ if and only if $\alpha = -\frac{1}{2}$. This gives the existence and the uniqueness of the tensor, as well as its exact expression. We now study the case of the degree two.

2.2 Ingredients for the tensor of degree two

The method   Many more terms will contain the tensor of degree two, because in this case, there are the possibilities of using the four indexed R_{ijkl}, with indices contracted, as in $R_{iabc} R_k{}^{abc}$ or as in $R_{iakb} R^{ab}$. So we first have to determine all possible terms, and then calculate all the constants appearing in the linear combination forming our tensor. We recall that we note Σ_{ik} for this tensor. Next, using the law of conservation of energy for Σ_{ik}, we show that there is a unique solution to this set of constants.

The general form of the tensor   To find the components of Σ_{ik} of degree two, we have simply to multiply two tensors of the form R_{abcd}, R_{ab}, or R and use as well g_{ik}, where the indices a, b, c, d are chosen to be i or k, or are otherwise contracted.

Products containing the scalar curvature R   For a product R^2 the only possible term is $R^2 g_{ik}$, for a product of R_{ab} with R, again one possibility, which is $R R_{ik}$.

Products Ricci-Ricci   For two products of R_{ab}, the indices i and k must belong to different R_{ab}, to avoid the appearance of the contraction R, a case already studied, and using that R_{ab} is symmetric, we get the only $R_{ia} R_k{}^a$.

Products Ricci-Riemann For products of R_{ab} and R_{abcd}, the term R_{ik} cannot appear, otherwise the contraction of R_{abcd} is R. As well, if R_{ia} appears, using the symmetries in the indices of R_{abcd}, we can suppose that k is the first index. We have then an expression of the form $R_{ia}R_{kpqr}$, where, among the indices, two are in the up position, one in the down position, a appears once, in the up position, to be contracted with the index a of R_{ia}, and say b appears twice among p, q and r, and is contracted. Then, in this Riemann tensor a cannot be the second index, otherwise the contraction over b is zero, so we can suppose a is the third index, and the contraction over b gives us another Ricci. So nor i neither k can appear in the Ricci, and we have then an R^{ab} where a, b are to be contracted with indices of a Riemann tensor. As R^{ab} is symmetric in a, b, it cannot be contracted with indices a, b placed in an antisymmetric position in R_{pqcd}, and as R_{pqcd} is antisymmetric in the first two indices and also in the last two, there is one a in the first two and one b in the last two. Using again the symmetries of the indices in the Riemann tensor, we can chose i in first place and k in the third, and we are left with the only possibility $R^{ab}R_{iakb}$.

Products Riemann-Riemann For the product of two Riemann tensors, it is quite direct to see that the only possibility is $R_i{}^{abc}R_{kabc}$. First, as before, we can suppose that i is the first index of the first Riemann. Now if i and k appear only in the first Riemann, c for example appears twice in the second, giving us zero or Ricci. So k is the first index of the second Riemann. Now we can chose the first as $R_i{}^{abc}$ and using the antisymmetry of the second tensor in the last two indices, we can suppose that in it, the last two indices are in alphabetical order. We are left with R_{kabc}, R_{kbac} and R_{kcab}. Using now that in the first Riemann b and c appear in antisymmetric positions, we have $R_i{}^{abc}R_{kbac} = -R_i{}^{abc}R_{kcab}$. Using finally the identity $R_{kabc} - R_{kbac} + R_{kcab} = 0$, we see that all possible tensors can be written only in terms of $R_i{}^{abc}R_{kabc}$.

Terms involving the metric tensor In all this, we have discarded the possibility of the appearance of g_{ik}, but the same arguments permit to conclude that the only possible terms are $R^{(4)}g_{ik}$ where $R^{(4)} = R^{abcd}R_{abcd}$, $R^{(2)}g_{ik}$ where $R^{(2)} = R^{ab}R_{ab}$ and of course the $R^2 g_{ik}$ first considered.

Synthesis   To conclude we get then the most general tensor Σ_{ik} of degree two :

$$\Sigma_{ik} = R_i{}^{abc}R_{kabc} + \alpha R_{iakb}R^{ab} + \beta R_{ia}R_k{}^a + \gamma R_{ik}R + \left(\delta R^{(4)} + \epsilon R^{(2)} + \eta R^2\right)g_{ik} \qquad (2.1)$$

2.3   Three formulas

In order to calculate the coefficients appearing in $\nabla^i\Sigma_{ik}$, we need a first formula :

$$\nabla^i R_{iabc} = \nabla_b R_{ac} - \nabla_c R_{ab} \qquad (2.2)$$

Starting with the Bianchi identity :

$$\nabla_m R_{nabc} + \nabla_c R_{namb} + \nabla_b R_{nacm} = 0 \qquad (2.3)$$

and contracting over m and n, we obtain (24.2) directly :

$$\nabla^n R_{nabc} + \nabla_c R_{ab} - \nabla_b R_{ac} = 0$$

Here, we adopt the convention that the contraction of the first and the third indices in the Riemann tensor gives the Ricci tensor, and then the contraction of the first and fourth indices in the Riemann tensor gives minus the Ricci tensor, because of the antisymmetry of third and fourth indices in the Riemann tensor. This gives the formula (2.2). Now, we calculate the coefficients of $\nabla^i\Sigma_{ik}$ one by one. First we need a second formula :

$$\nabla^i(R_i{}^{abc}R_{kabc}) = (\nabla_b R_{ac})R_k{}^{abc} - (\nabla_c R_{ab})R_k{}^{abc} + 2R^{iabc}(\nabla_c R_{kabi}) \qquad (2.4)$$

Using the properties of the connection ∇, we find :

$$\nabla^i(R_i{}^{abc}R_{kabc}) = (\nabla^i R_i{}^{abc})R_{kabc} + R_i{}^{abc}(\nabla^i R_{kabc}) \qquad (2.5)$$

and using

$$(\nabla^i R_i{}^{abc})R_{kabc} = (\nabla^i R_{iabc})R_k{}^{abc} \qquad (2.6)$$

as well as equation (2.2), we obtain immediately that the first term of the right hand side of (2.5) equals the first two terms of formula (2.2). So, we only need to prove that the

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second term on the right hand side of (2.5) equals the third of formula (2.4). Now, using the Bianchi identity (2.3), we have :

$$R_i{}^{abc}(\nabla^i R_{kabc}) = -R_i{}^{abc}\nabla_b R_{kac}{}^i - R^{iabc}\nabla_c R_{kaib} \tag{2.7}$$

Using the antisymmetry of the indices b and c in the first Riemann tensor of the term $-R_i{}^{abc}\nabla_b R_{kac}{}^i$, we obtain that this term equals $R_i{}^{abc}\nabla_c R_{kab}{}^i$, which also equals the term $-R^{iabc}\nabla_c R_{kaib}$, using the antisymmetry of i and b in the second Riemann tensor. Altogether, we see that formula (2.4) has been proved. Now, we study the term arising in $\nabla^i \Sigma_{ik}$ from the second term of Σ_{ik}, asserting the following formula :

$$\nabla^i(\alpha R_{iakb} R^{ab}) = \alpha R^{ab}(\nabla_k R_{ab}) - \alpha R^{ab}(\nabla_b R_{ak}) - \alpha(\nabla_c R_{ab})R_k{}^{abc} \tag{2.8}$$

To prove it, we use :

$$\nabla^i(R_{iakb} R^{ab}) = (\nabla^i R_{iakb})R^{ab} + R_{iakb}\nabla^i R^{ab}$$

$$= (\nabla^i R_{iakb})R^{ab} + R^{ia}{}_k{}^b \nabla_i R_{ab} \tag{2.9}$$

Now, applying to the first term on the right hand side of (24.9) the identity (2.2), we readily obtain the first two terms on the right hand side of formula (2.8). But the second term on the right hand side of (2.9) can be written $(\nabla_c R_{ab})R^{ca}{}_k{}^b$ and since exchanging the two pairs of indices in the Riemann tensor does not change its value, we obtain $(\nabla_c R_{ab})R_k{}^{bca}$. Now, using the well known identity

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0 \tag{2.10}$$

this term becomes

$$-(\nabla_c R_{ab})R_k{}^{abc} - (\nabla_c R_{ab})R_k{}^{cab}$$

In this last equation, the second term gives zero because a, b are contracted and appear in symmetric positions in the Ricci tensor and in antisymmetric positions in the Riemann tensor. This finishes the proof of formula (2.8).

2.4 Computing the coefficients of the tensor

Taking the β and γ terms of Σ_{ik}, and remembering that $\nabla^i R_{ik} = \frac{1}{2}\partial_k R$, we find at once :

$$\nabla^i(\beta R_{ia}R_k{}^a) = \frac{1}{2}\beta(\partial^a R)R_{ak} + \beta R^{ia}\nabla_i R_{ka} \qquad (2.11)$$

and

$$\nabla^i(\gamma R_{ik}R) = \frac{1}{2}\gamma(\partial_k R)R + \gamma R_{ik}(\partial^i R) \qquad (2.12)$$

Computing α, β, γ Looking closely at our equations, we see that the second terms of (2.8) and (2.11) can be eliminated by the choice $\alpha = \beta$, and that the first term of (2.11) gives zero, when combined with the second term of (2.12), provided that we choose the relation $\beta = -2\gamma$. So, by simple inspection of our equations, we possess an easy way to calculate our coefficients.

A relation which simplifies the whole calculation Turning now to the δ-term, we have :

$$\nabla^i\left(R^{(4)}g_{ik}\right) = \nabla^i[R_{abcd}R^{abcd}g_{ik}] = 2(\nabla_k R_{abcd})R^{abcd}$$

which equals :

$$-2R^{abcd}\nabla_d R_{abkc} - 2R^{abcd}\nabla_c R_{abdk}$$

because of (2.3). Both of these terms equal $2R^{abcd}\nabla_c R_{abkd}$, the second because in the second Riemann tensor, d and k are in antisymmetric positions, and the first because in the first Riemann tensor, c and d are in antisymmetric positions. We thus find :

$$\delta\nabla^i\left(R^{(4)}g_{ik}\right) = \delta\nabla^i[R_{abcd}R^{abcd}g_{ik}] = 4\delta R^{abcd}\nabla_c R_{abkd}$$

$$= 4\delta\nabla_c(R^{abcd}R_{abkd}) - 4\delta R_{abkd}\nabla_c R^{abcd}$$

Now happens a considerable simplification, because the first term of the former equation can be written $4\delta\nabla^c(R_{abcd}R^{ab}{}_k{}^d)$. Thus, c, which is contracted, can be called i, and we can exchange the two pairs of indices in both Riemann tensors, obtaining : $4\delta\nabla^i(R_{idab}R_k{}^{dab})$. This term is exactly the term of $\nabla^i\Sigma_{ik}$ which corresponds to the first term in the sum giving Σ_{ik}. So we find that choosing $\delta = -\frac{1}{4}$, we eliminate all the terms of (2.4). We are

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now left with a very few terms, the first and the third terms on the right hand side of (2.8), the first term on the right hand side of (2.12), the last $-4\delta R_{abkd}\nabla_c R^{abcd}$ and finally the ϵ and η terms of (2.1).

Computation of the δ-term This term can be written

$$-4\delta(\nabla^c R_{abcd})R^{ab}{}_k{}^d = (\nabla_a R_{db} - \nabla_b R_{ad})R^{ab}{}_k{}^d$$

using the value of δ and also formula (2.2). The second term on the right hand side of this formula is equal to the first, because in the Riemann tensor, a and b appear in antisymmetric positions, and we are left with :

$$2(\nabla_a R_{bd})R^{ab}{}_k{}^d = 2(\nabla_a R_{bd})R_k{}^{dab} = 2(\nabla_c R_{ab})R_k{}^{bca}$$

Indeed, we obtain the first equality by exchanging the pairs of indices in the Riemann tensor, and the second by renaming contracted indices. We use formula (2.10) to write $R_k{}^{bca} = -R_k{}^{abc} - R_k{}^{cab}$, and we observe that the second term has a and b in antisymmetric positions, which gives zero because these indices are contracted with $\nabla_c R_{ab}$. So the calculation of the δ-term of $\nabla^i \Sigma_{ik}$ is finished, and gives us only $-2(\nabla_c R_{ab})R_k{}^{abc}$, this term vanishing with the third term of (2.8) if and only if $\alpha = -2$. Comparing this result with the other relations obtained for β and γ, we now find $\beta = -2$, and $\gamma = +1$.

The ϵ-term We are now ready to study the ϵ-term. We know that we still have to cancel the first term of (2.8) and the first term of (2.12).

$$\epsilon\nabla^i[R_{ab}R^{ab}g_{ik}] = 2\epsilon(\nabla_k R_{ab})R^{ab}$$

cancels directly the first term of (2.8) provided $2\epsilon = -\alpha$, so $\epsilon = +1$.

The η-term The η-term gives $\eta\nabla^i[R^2 g_{ik}] = 2\eta R(\partial_k R)$, cancelling the first term of (2.12) provided $2\eta = -\frac{1}{2}\gamma$, which leads to $\eta = -\frac{1}{4}$, providing us finally a set of constants for which $\nabla^i \Sigma_{ik} = 0$. Finally we proved the statement of existence in the following theorem :

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Theorem : There exists a unique tensor Σ_{ik}, constructed from all possible products of degree two of the Riemann tensor, its contractions, and the metric tensor, which verifies the law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. This tensor contains effectively all possible products and has the form :

$$\Sigma_{ik} = R_i{}^{abc} R_{kabc} - 2R_{iakb} R^{ab} - 2R_{ia} R_k{}^a + R_{ik} R - \frac{1}{4} \left(R^{(4)} - 4R^{(2)} + R^2 \right) g_{ik} \qquad (2.13)$$

where $R^{(4)} = R^{abcd} R_{abcd}$ and $R^{(2)} = R^{ab} R_{ab}$

Existence As we said, the existence in the theorem has been proved before, we just notice that we used for this proof all identities we know concerning the Riemann tensor and its contractions.

Uniqueness Of course, we have also proved that there was no more possible products which could be ingredients of the tensor Σ_{ik}, and that our coefficients formed the complete set of degrees of freedom of our mathematical problem. Finding these coefficients has been possible because we could cancel all terms in $\nabla^i \Sigma_{ik} = 0$, using the well known relations on the Riemann tensor. It appears that, as there does not exist any such other relation on this tensor, available in the generic situation, this was the unique manner of cancelling these terms, and that the coefficients of Σ_{ik} are unique. Here, we give a method to obtain an explicit proof of the uniqueness of Σ_{ik} : starting with the value of Σ_{ik} with all its coefficients, at first undetermined, we compute $\nabla^i \Sigma_{ik}$ in different explicit choices of the metric g_{ik}, and each example gives us a linear combination of our coefficients, that we put equal to zero. So we find a linear system for these coefficients and with enough choices of different g_{ik}, we obtain enough equations, to prove finally that only the coefficients of the theorem give zero in the generic situation.

3 Higher dimensions, topology and complex gravity

3.1 The conjecture in higher dimensions

From what has been done in the case of degree two, it is easily guessed what can be done as well in the case of degree n. We can consider a tensor Σ_{ik}, of degree n in the Riemann tensor and its contractions, and first find all possible products of degree n that could appear in Σ_{ik}. Then, we find all coefficients by imposing that in $\nabla^i \Sigma_{ik}$, all terms vanish. Looking at the case $n = 1$ and $n = 2$, it should be clear that it is a way of proving the following conjecture :

Conjecture : There is a unique tensor Σ_{ik} constructed from all possible products of degree n in the Riemann tensor and its contractions, constructed with the metric tensor too, and which verifies the law of conservation of energy : $\nabla^i \Sigma_{ik} = 0$. This tensor has the form :

$$\Sigma_{ik} = \tilde{\Sigma}_{ik} - \frac{1}{2n} \tilde{\Sigma} g_{ik}$$

where $\tilde{\Sigma}$ is the Euler form in dimension $2n$, as well as the trace of $\tilde{\Sigma}_{ik}$. Furthermore, Σ_{ik} vanishes, becomes it comes, using the calculus of variation, from the topological Euler lagrangian.

3.2 Topology

The appearance in the tensor of degree 2 of the Gauss-Bonnet term

$$\tilde{\Sigma} = R^{(4)} - 4R^{(2)} + R^2$$

authorizes us to conjecture that our tensor completely vanishes in dimensions four, because it comes from the topological Gauss-Bonnet action :

$$\int \sqrt{-g} (R^{(4)} - 4R^{(2)} + R^2)$$

In dimensions different from four, the same tensor, of degree two, comes from the would-be-a-Gauss-Bonnet action :

$$\int \sqrt{-g}(R^{(4)} - 4R^{(2)} + R^2)$$

We thus have found an interesting method to write, from an a priori trivial topological action, a non trivial equation : start from the topological action in dimension n, go to another dimension where the same action is not trivial anymore, and use the calculus of variation to extract the tensorial equation. Then, take the tensor, and go back to the critical dimension. The question is : does the tensor obtained in this way should be discarded as being trivial or is it relevant to describe some kind of physics? This has been the first route which we used to find our equation of quantum gravity. Because the gravitational tensor of degree 2 first displays a dimensionless coupling constant, and second fits so well with the energy-momentum tensors of the other interactions, even if it is identically zero, we though there should be some kind of physics behind. We finally retained the idea of keeping only its trace in the equation, which gave the Λ term, the law of conservation of energy being in the equation of quantum gravity being provided by the variations of $\kappa(\epsilon)$.

A dimensionless coupling constant   Forgetting that our tensor vanishes for one moment, we consider the equation that such a tensor would give :

$$\Sigma_{ik} = \kappa T_{ik}$$

To determine the dimension of the coupling constant, we look at :

$$\Sigma_0^0 = \kappa T_0^0$$

Here the Riemann and Ricci tensors, when containing the same number of up and down indices, as well as the scalar curvature, have dimension $[L]^{-2}$, where $[L]$ is a length. So, Σ_0^0 has dimension $[L]^{-4}$. Now, T_0^0 equals ϵ, the energy density of matter, and has dimension, in dimension $D = 4$, $[E][L]^{-3} \sim \hbar[T]^{-1}[L]^{-3} \sim \hbar c[L]^{-4}$, since energy $[E]$ has dimension $\hbar[T]^{-1}$ and where of course $[T]$ is a time. Comparing these two results, we see that

$$\kappa = \frac{\kappa_0}{\hbar c}$$

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where κ_0 has no dimension at all.

3.3 Complex gravity

We know that our tensor Σ_{ik} probably vanishes because it is the energy-momentum tensor coming from a topological lagrangian by the calculus of variations, but there is another form of this tensor, which at least at first sight, is not necessarily trivial, and which could prove itself very interesting. Because it is of second order in the curvature tensor, Σ_{ik} possesses a natural extension to complex gravity. As in the quantum tensors describing particles of different spin, we can write down a tensor of degree two by doubling the curvature terms by complex conjugates. By inspection of these known quantum tensors, we guess easily the procedure to follow. Indeed, we pose as new fundamental variables, the complex metric g_{ik} verifying the condition :

$$g_{ki}^* = g_{ik} \tag{3.1}$$

where z^* corresponds to the complex conjugate of z, and we pose the complex tensor :

$$\Sigma_{ik} = \frac{1}{2} R^*{}_i{}^{abc} R_{kabc} + \frac{1}{2} R_i{}^{abc} R^*_{kabc} - R^*{}_{iakb} R^{ab} - R_{iakb} R^{*ab} - R^*{}_{ia} R_k{}^a - R_{ia} R_k^{*a}$$

$$+ \frac{1}{2} R^*{}_{ik} R + \frac{1}{2} R_{ik} R^* - \frac{1}{4} \left(R^{(4)} - 4R^{(2)} + RR^* \right) g_{ik} \tag{3.2}$$

where $R^{(4)} = R^{*abcd} R_{abcd}$ and $R^{(2)} = R^{*ab} R_{ab}$. Equations (3.1) and (3.2) should normally imply that Σ_{ik} is a real symmetric tensor which verifies the condition of conservation of energy.

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